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1968

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Abstract

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UDC 517.511

MATHEMATICS

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ON ONE APPROACH TO THE CONCEPT OF LIMIT

(Presented by Academician A. N. Tikhonov, 10 VII 1968)

Our aim is to construct a theory of limits in such a way that every sequence has a limit.

Let S be the vector space consisting of all real countable sequences. We identify the real number a with the corresponding stationary sequence, i.e., with the sequence $x = \{x_n\}$ such that $x_n = a$ for every n . The set Γ of all sequences converging to zero is a subspace in S . The limit of a sequence x is equal to the number a if and only if the cosets $\Gamma + a$ and $\Gamma + x$ coincide. This circumstance allows us to regard the coset $\Gamma + x$ as the limit of the sequence $x = \{x_n\}$ from S : $\Gamma + x = \text{Lim } x_n$. We identify the real number a with the coset $\Gamma + a$. If the sequence x converges and $a = \lim x_n$, then $a = \Gamma + a = \Gamma + x$, i.e., the introduced definition of limit does not contradict the classical one. The coset $\Gamma + x$ exists for every x from S , i.e., every sequence has a limit $\Gamma \text{Lim } x_n$. On the basis of these preliminary considerations we shall construct a theory of limits. From the example considered it is clear that the limit $\Gamma \text{Lim } x_n$ of a sequence of real numbers may be not a real number, but an element of some set containing the number line.

Let I be a right-filtering partially ordered set, and let to each index $i \in I$ there correspond an (additive) group G_i ; denote the complete direct sum of these groups by $\Sigma^* G_i$. If all G_i coincide, then we denote $G_i = G$, $\Sigma^* G_i = G^I$. We shall call an I -indexed sequence an element $\{a_i\}$ of $\Sigma^* G_i$. Take a normal divisor H in the group $\Sigma^* G_i$. By the limit $H \text{Lim } a_i$ modulo H of an I -indexed sequence $a = \{a_i\}$ from $\Sigma^* G_i$ we shall mean the coset $H + a$. If all G_i are equal to one another and equal to G , then denote the factor group $(\Sigma^* G_i)/H$ by G_H^I . The limit of a sum is equal to the sum of the limits. If all G_i are vector spaces over a field K , and H is a subspace in the space $\Sigma^* G_i$, then $H \text{Lim } \alpha x_i = \alpha H \text{Lim } x_i$. If all G_i are rings and H is an ideal in the ring $\Sigma^* G_i$, then $H \text{Lim}(x_i y_i) = (H \text{Lim } x_i)(H \text{Lim } y_i)$.

We shall say that a normal divisor H of $\Sigma^* G_i$ separates a subset M of $\Sigma^* G_i$ if every coset modulo H contains no more than one element of M . If H separates M and the class $H + a$ contains an element m of M , then we identify the coset

$H + a$ with m . In particular, if M is the number line, $m \in M$, $m \in H \operatorname{Lim} a_i = H + a$, then we shall regard the limit $H \operatorname{Lim} a_i$ as the real number m . An I -indexed sequence $\{x_i\}$ from $\Sigma^* G_i$ will be called almost zero if there exists an index $i_0 \in I$ such that $x_i = 0$ as soon as $i \geq i_0$. The set T of all almost zero sequences is a normal divisor in the group $\Sigma^* G_i$. The limit $T \operatorname{Lim} x_i$ will be called the fine limit. The group $(\Sigma^* G_i)/T$ is the reduced product of the groups G_i . A sequence $\{a_i\}$ will be called stationary if there exists a from $G = \bigcap G_i$ such that $a_i = a$ for every i from I ; since T separates G , we can identify a with $T + \{a_i\}$. If all G_i are rings, then $\Sigma^* G_i$ is also a ring, and T is a two-sided ideal in the ring $\Sigma^* G_i$, therefore

$T \operatorname{Lim}(x_i y_i) = (T \operatorname{Lim} x_i)(T \operatorname{Lim} y_i)$. Let all G_i be vector structures. Put $\{x_i\} \cap \{y_i\} = \{x_i \cap y_i\}$, $\{x_i\} \cup \{y_i\} = \{x_i \cup y_i\}$. Then

$\Sigma^* G_i$ is a vector structure. If, moreover, H is an l -ideal in $\Sigma^* G_i$, then the factor group $(\Sigma^* G_i)/H$ is also a vector structure, and $H \operatorname{Lim}(x_i \cup y_i) = (H \operatorname{Lim} x_i) \cup (H \operatorname{Lim} y_i)$, $H \operatorname{Lim}(x_i \cap y_i) = (H \operatorname{Lim} x_i) \cap (H \operatorname{Lim} y_i)$, $|H \operatorname{Lim} x_i| = H \operatorname{Lim} |x_i|$, and from $x \geq y$ it follows that $H \operatorname{Lim} x_i \geq H \operatorname{Lim} y_i$. Suppose that each vector structure G_i contains the number line R as a vector substructure. A sequence $\{t_i\}$ from $\Sigma^* G_i$ will be called infinitesimal if, for any positive ε from R , there exists an index i_ε from I such that $|x_i| < \varepsilon$ as soon as $i \geq i_\varepsilon$. The set Γ of all infinitesimal sequences is an l -ideal in the vector structure $\Sigma^* G_i$. The limit $\Gamma \operatorname{Lim} x_i$ will be called the rough limit. Γ separates R .

Denote by N the set of all natural numbers, and by R^m the real m -dimensional vector space with fixed basis $e_1, \dots, e_m$. For any n from N , define the set A_n^m (a half-open cube), assigning a vector $x = (x_1, \dots, x_m)$ from R^m to A_n^m if $-n \leq x_i < n$ ($1 \leq i \leq m$). For any n from N and any x from R , there exists a unique number $r_n(x)$ from R such that $-n \leq r_n(x) < n$ and $x \equiv r_n(x) \pmod{2n}$, i.e. the difference $x - r_n(x)$ is an integer multiple of $2n$. We have defined the function $r_n(x)$ on R with values in A_n^1 . Construct the mapping $r_n^m(x)$ of the space R^m onto the half-open cube A_n^m . For any $x = (x_1, \dots, x_m)$, put $r_n^m(x) = (r_n(x_1), \dots, r_n(x_m))$. Introduce on the cube A_n^m the operation $\oplus$, setting, for any x, y from A_n^m , $x \oplus y = r_n^m(x + y)$. A_n^m is a group. On A_n^m the natural topology is introduced, and A_n^m is a bicomact topological group (the m -dimensional torus). Lebesgue measure on the cube A_n^m is Haar measure. Real measurable functions on the cube A_n^m are regarded, as usual, as equal if they coincide almost everywhere. Let F_n^m be the set of all real bounded measurable functions on A_n^m . The thin limit

$$f(x) = T \operatorname{Lim}_n f_n(x)$$

from $(\Sigma^* F_n^m)/T$ will be called a generalized function on R^m .

A real function $\varphi(x_1, \dots, x_k)$ on R^k will be called locally bounded if it is bounded on every bounded subset of R^k . Let $\varphi(x_1, \dots, x_k)$ be a Borel locally bounded function on R^k , and let generalized functions

$$f_1(x) = T \operatorname{Lim}_n f_n^1(x), \dots, f_k(x) = T \operatorname{Lim}_n f_n^k(x)$$

be given on R^m . Define the generalized function $\varphi(f_1(x), \dots,$

$$\dots, f_k(x)),$$

putting

$$\varphi(f_1(x), \dots, f_k(x)) = T \operatorname{Lim}_n \varphi(f_n^1(x), \dots, f_n^k(x)).$$

Since the function $\varphi(x, y) = xy$ on R^2 is Borel and locally bounded, the product of any pair of generalized functions on R^m is defined.

Let a series

$$\sum_{i=1}^{\infty} a_i x^i$$

and a generalized function $f(x) = T \operatorname{Lim}_n f_n(x)$ on R^m be given. The sum

$$\sum_{i=1}^{\infty} a_i f^i(x) = T \operatorname{Lim}_n \sum_{i=1}^n a_i f_n^i(x)$$

is also a generalized function on R^m . Let ψ be a measurable mapping of the space R^m into itself, i.e. for any measurable subset E of R^m its preimage is measurable. For any n from N , define the mapping ψ_n of the cube A_n^m into itself by putting

$$\psi_n(x) = r_n^m(\psi(x)).$$

If $f(x) = T \operatorname{Lim}_n f_n(x)$ is a generalized function on R^m , then, for any n from N , $f_n(\psi_n(x)) \in F_n^m$. The thin limit $T \operatorname{Lim}_n f_n(\psi_n(x))$ is a generalized function. Denote

$$f(\psi(x)) = T \operatorname{Lim}_n f_n(\psi_n(x)).$$

Let $f(x) = T \operatorname{Lim}_n f_n(x)$ be a generalized function on R^m , i an integer, $1 \leq i \leq m$. For any n from N , the measurable function on A_n^m

$$\partial_n f_n(x) / \partial x_i = n(f_n(x \oplus n^{-1} e_i) - f_n(x))$$

is bounded, i.e. the limit $T \operatorname{Lim}_n (\partial_n f_n(x) / \partial x_i)$ is a generalized function on R^m . Denote

$$\partial f / \partial x_i = T \operatorname{Lim}_n (\partial_n f_n(x) / \partial x_i).$$

For $m = 1$ we denote

$$\partial f(x) / \partial x = df(x) / dx = f'(x).$$

For any n from N , define the cube V_n^m as the set of all vectors from R^m such that $0 < x_1 < n^{-1}, \dots, 0 < x_m < n^{-1}$. Denote by Φ_n^m the set of all real bounded measurable functions on the cube V_n^m , $\Phi^m = (\sum_n^* \Phi_n^m) / T$, $A^m = (\sum_n^* A_n^m) / T$. Let $f(x) = T \operatorname{Lim}_n f_n(x)$ be a generalized function on R^m . We associate with it a function $f^*(x)$ on A^m . Take a point $\alpha = T \operatorname{Lim}_n \alpha_n$ from A^m and construct the value $f^*(\alpha)$. For any n from N , construct a function $f_n(t_n, \alpha_n)$ on V_n^m , which we shall regard as a function of t_n , setting, for any t_n from V_n^m ,

$f_n(t_n, \alpha_n) = f_n(t_n \oplus \alpha_n)$. The limit $T \operatorname{Lim}_n f_n(t_n, \alpha_n)$ from Φ^m will be regarded as the value of the function $f^*(x)$ at the point α and denoted $f^*(T \operatorname{Lim}_n \alpha_n) = T \operatorname{Lim}_n f_n(t_n, \alpha_n)$. If $f(x) = T \operatorname{Lim}_n f_n(x)$, $g(x) = T \operatorname{Lim}_n g_n(x)$ are generalized functions on R^m and, for any $\alpha = T \operatorname{Lim}_n \alpha_n$ from A^m , $f^*(\alpha) = g^*(\alpha)$, then the generalized functions $f(x)$ and $g(x)$ coincide. Therefore we may identify a generalized function $f(x)$ on R^m with the function $f^*(x)$ on A^m . In this sense every generalized function $f(x)$ on R^m is an ordinary function on A^m with values in Φ^m . To each generalized function $f(x) = T \operatorname{Lim}_n f_n(x)$ on R^m there correspond the integrals

$$T \int f d\mu = T \operatorname{Lim}_n \int_{A_n^m} f_n(x) d\mu \quad \text{and} \quad \Gamma \int f d\mu = \Gamma \operatorname{Lim}_n \int_{A_n^m} f_n(x) d\mu.$$

Let $\varphi(x)$ be a real measurable locally bounded function on R^m . Denote

$$\begin{aligned} T \int f(x) \varphi(x) d\mu &= T \operatorname{Lim}_n \int_{A_n^m} f_n(x) \varphi(x) d\mu, & \Gamma \int f(x) \varphi(x) d\mu \\ &= \Gamma \operatorname{Lim}_n \int_{A_n^m} f_n(x) \varphi(x) d\mu. \end{aligned}$$

Let $f(x) = T \operatorname{Lim}_n f_n(x)$ and $g(x) = T \operatorname{Lim}_n g_n(x)$ be generalized functions on R^m .

Denote

$$h(x) = f(x) * g(x) = T \operatorname{Lim}_n \int_{A_n^m} f_n(\xi) g_n(x - \xi) d\xi,$$

and call the generalized function thus obtained the convolution. All the preceding operations are also applicable to complex generalized functions $f(x) = T \operatorname{Lim}_n (\alpha_n(x) + i\beta_n(x))$. Let $f(x) = T \operatorname{Lim}_n f_n(x)$ be a complex generalized function on R^m . For any n from N and any $\lambda = (\lambda_1, \dots, \lambda_m)$ from A_n^m , denote

$$c_n(\lambda) = \int_{A_n^m} e^{-2\pi i \lambda x} f_n(x) dx,$$

where $\lambda x = \lambda_1 x_1 + \dots + \lambda_m x_m$.

The generalized function $c(\lambda) = T \operatorname{Lim}_n c_n(\lambda)$ will be called the Fourier transform of the generalized function $f(x) = T \operatorname{Lim}_n f_n(x)$.

Let $\varphi(x_1, \dots, x_{k+2})$ be a locally bounded Borel function on R^{k+2} , and let generalized functions $f_1(x) = T \operatorname{Lim}_n f_n^1(x), \dots, f_k(x) = T \operatorname{Lim}_n f_n^k(x)$ be given on R .

We shall say that the generalized function $y(x) = T \operatorname{Lim}_n y_n(x)$ is a solution of the differential equation

$$dy(x)/dx = \varphi(x, y(x), f_1(x), \dots, f_k(x)) \quad (1)$$

on the interval $a \leq x \leq b$, if for every x from this interval relation (1) is satisfied. Choose a point $y_0 = T \operatorname{Lim}_n y_n^0(t)$ from Φ^m , and we shall seek a generalized function $y(x) = T \operatorname{Lim}_n y_n(x)$ on R satisfying relation (1) for $a \leq x \leq b$ and the initial condition $y(a) = y_0$. Choose n_0 from N so that $n_0 > |a| + 1$ and $n_0 > |b| + 1$. For $n < n_0$ put $y_n(x) = 0$. Let $n \geq n_0$. For $-n < x < a$ put $y_n(x) = 0$. For $a < x < a + n^{-1}$ put $y_n(x) = y_n^0(x - a)$. For $a + n^{-1} < x < a + 2n^{-1}$ put

$$y_n(x) = y_n(x - n^{-1}) + n\varphi(x - n^{-1}, y(x - n^{-1}), f_1(x - n^{-1}), \dots, f_k(x - n^{-1})). \quad (2)$$

For $a + 2n^{-1} < x < a + 3n^{-1}$ we define $y_n(x)$ by formula (2), and so on. There exists $k_n \in N$ such that

$$a + n^{-1}k_n \leq b < a + n^{-1}(k_n + 1).$$

For $a + n^{-1}(k_n + 1) < x < n$ we put $y_n(x) = 0$. The generalized function

$$y(x) = \Gamma \operatorname{Lim} y_n(x)$$

is the unique solution of the problem posed.

Suppose generalized functions $f(x)$ and $g(x)$ are given on R . The equation

$$y'(x) = f(x)g(x)y(x) \quad (3)$$

is a special case of equation (1). We note that similar equations (with a four-dimensional argument x) are used in quantum electrodynamics¹, and that, in the known presentations of the theory of generalized functions, solving equation (3) is connected with difficulties.

Let $f(x, y)$ be a Borel function on R^2 , let $y(x)$ be an ordinary solution of the equation $y' = f(x, y)$ on the interval $a \leq x \leq b$, suppose that the Lipschitz condition is satisfied for all y and z , and that along the curve $y = y(x)$, for $a \leq x \leq b$, the derivative $df(x, y(x))/dx$ is bounded. If

$$z(x) = \Gamma \operatorname{Lim} z_n(x)$$

is a generalized solution of the equation $y' = f(x, y)$, then the sequence $\{z_n(x)\}$ converges uniformly on the interval $a \leq x \leq b$ to the function $y(x)$.

Put $e(x) = 0$ for $-n < x < 0$, $e(x) = 1$ for $0 < x < n$; $\delta(x) = e'(x)$. If $\varphi(x)$ is a measurable function on R and, for large n , the integrals

$$\int_{-n^{-1}}^0 \varphi(x) dx \quad \text{and} \quad \int_{n-n^{-1}}^n \varphi(x) dx$$

exist, then

$$\Gamma \int \delta(x) \varphi(x) dx = \Gamma \operatorname{Lim}_n n \int_{-n^{-1}}^0 \varphi(x) dx - \Gamma \operatorname{Lim}_n n \int_{n-n^{-1}}^n \varphi(x) dx,$$

$$\Gamma \int \delta^2(x) \varphi(x) dx = \Gamma \operatorname{Lim}_n n^2 \int_{-n^{-1}}^0 \varphi(x) dx + \Gamma \operatorname{Lim}_n n^2 \int_{n-n^{-1}}^n \varphi(x) dx.$$

If the finite function $\varphi(x)$ has a finite first derivative at zero, then

$$\Gamma \int \delta^2(x) \varphi(x) dx = \varphi(0) \Gamma \operatorname{Lim}_n n - \frac{1}{2} \varphi'(0).$$

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Received
6 VII 1967

References Cited

¹ P. K. Rashevskii, UMN, **13**, no. 3 (81), 3 (1958).

Note: Figure translations are in progress. See original paper for figures.

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