

ON THE SPECTRAL REPRESENTATION OF SPACES OF CLOSED SUBSETS OF BICOMPACTA

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Abstract

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MATHEMATICS

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ON THE SPECTRAL REPRESENTATION OF SPACES OF CLOSED SUBSETS OF BICOMPACTA

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The space of closed subsets of a given topological space X in the Vietoris topology is called the space $\exp X$, whose points are the closed subsets H of the space X , and whose basic open sets are all sets of the form $\langle U_{\alpha(1)}, \dots$

$$\dots, U_{\alpha(n)} \rangle = \left\{ H = [H] \subset X : H \subset \bigcup_{i=1}^n U_{\alpha(i)}, H \cap U_{\alpha(i)} \neq \emptyset, i = 1, \dots, n \right\},$$

where $U_{\alpha(1)}, \dots, U_{\alpha(n)}$ are open subsets of the space X . As is known ⁽²⁾, $\exp X$ is bicomact if and only if X is bicomact.

The main result of the paper is Theorem 1, which gives a spectral representation for the space of closed subsets of the limit of an inverse topological spectrum, and Theorem 3, in which the dyadicity of the space of closed subsets of a bicomactum is proved in the case when it itself is dyadic and its weight does not exceed $\aleph_1$.

In what follows, the point of the space $\exp X$ corresponding to a closed subset H of the space X will be denoted by the symbol $\widehat{H}$.

Lemma 1. *If $\varphi : X \rightarrow Y$ is a continuous mapping of the space X into the space Y , then the mapping $\psi : \exp X \rightarrow \exp Y$, defined as follows: $\psi(\widehat{H}) = \widehat{\varphi(H)}$, is a continuous mapping; moreover, $\psi(\exp X) = \exp \varphi(X)$, and, if the mapping φ is open, then the mapping ψ is also open.*

In what follows, the mapping ψ generated by the mapping φ by means of the construction described above will be called the exponential continuation of the mapping φ .

Theorem 1. *If $\{X_\alpha; \pi_\beta^\alpha\}$ is a spectrum, where the X_α are bicomacta, then $\{\exp X_\alpha; \omega_\beta^\alpha\}$ is also a spectrum, and moreover*

$$Y = \lim_{\leftarrow} \{\exp X_\alpha; \omega_\beta^\alpha\} = \exp \lim_{\leftarrow} \{X_\alpha; \pi_\beta^\alpha\},$$

where ω_β^α is the exponential continuation of the mapping π_β^α .

Proof. We shall first prove that $\{\exp X_\alpha; \omega_\beta^\alpha\}$ is a spectrum. The set $\{\alpha\}$ is directed, and it remains to show that if $\alpha > \beta > \gamma$, then $\omega_\gamma^\beta \omega_\beta^\alpha = \omega_\gamma^\alpha$. But for every $H = [H] \subset X$, evidently,

$$\omega_\gamma^\beta \omega_\beta^\alpha(\widehat{H}) = \omega_\gamma^\beta \widehat{\pi_\beta^\alpha(H)} = \widehat{\pi_\gamma^\beta \pi_\beta^\alpha(H)} = \widehat{\pi_\gamma^\alpha(H)} = \omega_\gamma^\alpha(\widehat{H}),$$

whence the required assertion follows.

As is known ⁽³⁾, if for the bicomactum $\exp X$ there exists a family $\{\varphi_\alpha\}$ of its mappings onto the spaces $\exp X_\alpha$, with $\omega_\beta^\alpha \varphi_\alpha = \varphi_\beta$, then it generates a continuous mapping φ of the bicomactum $\exp X$ onto Y . Denote by π_α the natural projections

$$X = \lim_{\leftarrow} \{X_\alpha; \pi_\beta^\alpha\}$$

onto X_α , and let $\varphi_\alpha : \exp X \rightarrow \exp X_\alpha$ be the exponential continuation of the mapping π_α . We now prove that $\omega_\beta^\alpha \varphi_\alpha = \varphi_\beta$. For $\alpha > \beta$ we have

$$\omega_\beta^\alpha \varphi_\alpha(\widehat{H}) = \omega_\beta^\alpha \widehat{\pi_\alpha(H)} = \widehat{\pi_\beta^\alpha \pi_\alpha(H)} = \widehat{\pi_\beta(H)} = \varphi_\beta(\widehat{H}).$$

Thus, for the space $\exp X$ we have obtained a family of mappings satisfying the required conditions, which generates a continuous mapping φ onto Y ,

and, by the bicomactness of $\exp X$, to prove the homeomorphism it suffices to show that φ is a condensation. Let $\widehat{H}_1 \in \exp X$, $\widehat{H}_2 \in \exp X$, $\widehat{H}_1 \neq \widehat{H}_2$. Obviously,

$$H_i = \lim_{\leftarrow} \{\pi_\alpha(H_i); \pi_\beta^\alpha|_{\pi_\alpha(H_i)}\},$$

where $\pi_\beta^\alpha|_{\pi_\alpha(H_i)}$ is the restriction of the mapping π_β^α to the set $\pi_\alpha(H_i)$, $i = 1, 2$, and, since $H_1 \neq H_2$, for some α we shall have $\pi_\alpha(H_1) \neq \pi_\alpha(H_2)$, or, what is the same, $\varphi_\alpha(\widehat{H}_1) \neq \varphi_\alpha(\widehat{H}_2)$. But this means that $\omega_\alpha \varphi(\widehat{H}_1) \neq \omega_\alpha \varphi(\widehat{H}_2)$, whence it follows that $\varphi(\widehat{H}_1) \neq \varphi(\widehat{H}_2)$. Hence φ is a condensation and a homeomorphism. The theorem is proved.

Lemma 2. *There exists a topological embedding of the Cantor discontinuum C into the interval $[0, 1]$ of the real line such that for any three points x, y, z of C , if $x \neq y$, then always $|x - z| \neq |x - y|$.*

In what follows the Cantor discontinuum will always be regarded as a subset of the line, situated in the indicated way.

Let H be a closed subset of C . Denote

$$D_0(H) = \{x \in H : 2x \leq \inf H + \sup H\}$$

and, correspondingly,

$$D_1(H) = \{x \in H : 2x \geq \inf H + \sup H\}.$$

In view of the remark made above, obviously,

$$D_0(H) \cap D_1(H) = \emptyset.$$

Lemma 3. *In the Hausdorff metric $*$ the relation $D_i(H)$ is continuous, $i = 0, 1$, i.e. if $\rho_h(H_1, H_2) \rightarrow 0$, then always $\rho_h(D_i(H_1), D_i(H_2)) \rightarrow 0$.*

Denote by AC the set of all perfect subsets of C , and we shall regard AC as a subset of $\exp C$.

Lemma 4. *There exists a continuous mapping f of the product $C \times AC$ onto C such that: 1) $f(C \times \{\hat{H}\}) = H$, where $\hat{H} \in AC$, and 2) the restriction of f to the set $C \times \{\hat{H}\}$ is a homeomorphism.*

Here we shall carry out the construction of the mapping f , and, for the sake of simplicity, omit the proof of the properties required by the lemma. If H is a perfect subset of C , then, obviously, $D_i(H)$ is also perfect, and for each sequence of zeros and ones $\{i(1), \dots, i(n)\}$ one can uniquely define $D_{i(1)\dots i(n)}(H)$ by the rule

$$D_{i(1)\dots i(n)}(H) = D_{i(n)}(D_{i(1)\dots i(n-1)}(H)).$$

It is not difficult to show that all sets of this form constitute a base of H , and each point $x \in H$ determines a sequence $I(x, H) = \{i(j)\}_{j=1}^{\infty}$ of zeros and ones, uniquely determined by the relations

$$D_{i(1)}(H) \supset D_{i(1)i(2)}(H) \supset \dots \ni x,$$

and conversely, each sequence $I = \{i(j)\}_{j=1}^{\infty}$ determines a point $x(I, H)$ satisfying the equality $I(x(I, H)) = I$. The mapping f is defined in the following way:

$$f(x, \hat{H}) = x(I(x, C), H).$$

With the aid of Lemma 3 it is proved that f satisfies the required conditions.

Definition. A mapping φ of a bicom pactum X onto a bicom pactum Y is called a d -mapping if φ is continuous and open and there exists a topological embedding of X into $C \times Y$ such that $\varphi = \pi_y|X$, where $\pi_y|X$ is the restriction of the projection of the product to the set X .

Lemma 5. *If φ is a d -mapping of the bicom pactum X onto the bicom pactum Y , and the sets $\varphi^{-1}(y)$ are perfect for every $y \in Y$, then there exists a homeomorphism $g : C \times Y \rightarrow X$ such that $\pi_y = \varphi g$.*

Proof. By the definition of a d -mapping there exists a topological embedding of X into $C \times Y$ such that $\varphi = \pi_y|X$. Define g then as follows:

$$g(c, y) = f(c, \pi_c \varphi^{-1}(y), y),$$

where f is a mapping satisfying the conditions of the preceding lemma. It is not difficult to see that g is one-to-one and, moreover, since $g(c, y) \in X$, we have

$$\varphi g(c, y) = \pi_y|X(g(c, y)) = y = \pi_y(c, y).$$

Let us now prove the continuity of the mapping g . Denote $H = \pi_c \varphi^{-1}(y)$. Let $V_\varepsilon \times W$ be some basic

* Recall the definition of the Hausdorff metric ρ_h : if H_1 and H_2 are closed subsets of a metric compactum X , then

$$\rho_h(H_1, H_2) = \max_{x_1 \in H_1} \min_{x_2 \in H_2} \rho(x_1, x_2) + \max_{x_1 \in H_2} \min_{x_2 \in H_1} \rho(x_1, x_2).$$

The topology induced by the Hausdorff metric in $\exp X$, for compact X , coincides with the Vietoris topology ⁽²⁾.

neighborhood of the point $g(c, y)$. Then, by Lemma 4, one can choose a neighborhood $\langle U_1, \dots, U_n \rangle$ of the point $\tilde{H}$ in $\exp C$ and a neighborhood U of the point c in C such that as soon as $\tilde{H}_1 \in \langle U_1, \dots, U_n \rangle$ and $c_1 \in U$, then

$$\rho(f(c, \tilde{H}), f(c_1, \tilde{H}_1)) < \varepsilon.$$

From the definition of the neighborhood $\langle U_1, \dots, U_n \rangle$ it follows that $H \subset \bigcup_{i=1}^n U_i$, $H \cap U_i \neq \emptyset$, $i = 1, \dots, n$. But then $C \times U_i \cap \varphi^{-1}(y) \neq \emptyset$ and is open, and

$$O_i = \varphi(S \times U_i \cap \varphi^{-1}(y)) \ni y$$

and is also open by virtue of the openness of the mapping φ . On the other hand,

$$O_0 = Y \setminus \varphi(X \setminus (C \times \bigcup_{i=1}^n U_i))$$

is also open and contains y . Consider now the neighborhood $U \times (O \cap W)$ of the point (c, y) , where $O = \bigcap_{i=1}^n O_i$. Then, if $(c_1, y_1) \in U \times (O \cap W)$, then

$$\pi_c \varphi^{-1} \varphi(c_1, y_1) = \pi_c \varphi^{-1}(y_1) \in \langle U_1, \dots, U_n \rangle, \quad c_1 \in U,$$

and, by the choice of the neighborhoods $\langle U_1, \dots, U_n \rangle$ and U , we have

$$\rho(f(c, \pi_c \varphi^{-1}(y)), f(c_1, \pi_c \varphi^{-1}(y_1))) < \varepsilon,$$

i.e.

$$f(c_1, \pi_c \varphi^{-1}(y_1)) \in V_\varepsilon, \quad y_1 \in W,$$

and

$$g(c_1, y_1) = (f(c_1, \pi_c \varphi^{-1}(y_1)), y_1) \in V_\varepsilon \times W,$$

which is what had to be proved.

Lemma 6. *If φ is a d -mapping of a bicomcompactum X onto a bicomcompactum Y , then there exists a continuous mapping ψ of the product $C \times Y$ onto X , with $\pi_y = \varphi\psi$, and ψ open.*

Proof. It is not difficult to see that the superposition of two d -mappings is again a d -mapping. Denoting by π_x the projection of the product $C \times X$ onto the factor X , we have that $\varphi\pi_x$ is a d -mapping, and for each $y \in Y$ the set $\pi_x^{-1}\varphi^{-1}(y)$ is perfect. Then, by Lemma 5, there exists a homeomorphism, and hence an open mapping,

$$g : C \times Y \rightarrow C \times X,$$

with $\pi_y = \varphi\pi_x g$. But then $\psi = \pi_x g$ is the required mapping, since $\pi_y = \varphi\psi$, and ψ is open as a superposition of two open mappings. The lemma is proved.

Lemma 7. *If φ is a d -mapping of a bicomcompactum X onto a bicomcompactum Y and ψ is a continuous mapping of a bicomcompactum Z onto Y , then there exists a mapping g of the product $C \times Z$ onto X such that $\psi\pi_z = \varphi g$; moreover, if ψ is an open mapping, then g can also be made open, and if φ is a d -mapping with perfect preimages for each $y \in Y$ and ψ is a homeomorphism, then g can also be made a homeomorphism.*

The proof follows immediately from the two preceding lemmas.

Theorem 2. *If $\{\alpha\}$ is a well-ordered set of indices and $\{X_\alpha, \pi_\beta^\alpha\}$ is an inverse spectrum such that: 1) X_1 is a dyadic bicomcompactum; 2) for every limit α one has*

$$X_\alpha = \lim_{\leftarrow} \{X_\beta; \pi_\gamma^\beta\}_{\beta < \alpha};$$

3) $\pi_\alpha^{\alpha+1}$ is a d -mapping for all α , then

$$X = \lim_{\leftarrow} \{X_\alpha; \pi_\beta^\alpha\}$$

is a dyadic bicomcompactum.

Proof. To each index α we associate its own copy C_α of the Cantor discontinuum. Next, putting

$$Y_\alpha = X_1 \times \prod_{\beta < \alpha} C_\beta$$

and taking ω_β^α to be the natural projection of the product onto the subproduct, we may consider the transfinite spectrum $\{Y_\alpha; \omega_\beta^\alpha\}$. Obviously,

$$\omega_\gamma^\beta \omega_\beta^\alpha = \omega_\gamma^\alpha,$$

and for limit α one has

$$Y_\alpha = \lim_{\leftarrow} \{Y_\beta; \omega_\gamma^\beta\}_{\beta < \alpha}.$$

To prove the dyadicity of X , it is enough to construct a continuous mapping

$$Y = \lim_{\leftarrow} \{Y_\alpha; \omega_\beta^\alpha\},$$

which is a dyadic bicom pactum, onto X , for which in turn it is enough to find a family of continuous mappings, commuting with the spectral ones, which, by known theorems ⁽³⁾, will generate a continuous mapping of the limit onto the limit. We shall construct the desired family by the method of transfinite induction. As $\varphi_1 : Y_1 \rightarrow X_1$ take the identity mapping. Suppose now that for all $\beta < \alpha$ mappings φ_β have been constructed, and for every $\gamma < \beta < \alpha$ the relation

$$\pi_\gamma^\beta \varphi_\beta = \varphi_\gamma \omega_\gamma^\beta$$

holds. Then, in the case when α is a limit transfinite, earlier

the constructed mappings induce a continuous mapping of the limit, and, since by the hypothesis of the theorem

$X_\alpha = \lim_{\leftarrow} \{X_\beta; \pi_\gamma^\beta\}_{\beta < \alpha}$ and, by construction,

$Y_\alpha = \lim_{\leftarrow} \{Y_\beta; \omega_\gamma^\beta\}_{\beta < \alpha}$, we have thereby obtained the required mapping φ_α .

If, however, α is an indeterminate transfinite number, then $\alpha = (\alpha - 1) + 1$, and by Lemma 7 there is a mapping φ_α such that $\pi_{\alpha-1}^\alpha \varphi_\alpha = \varphi_{\alpha-1} \omega_{\alpha-1}^\alpha$, and consequently φ_α is the required one. Thus a family of mappings has been found which generates a mapping continuous from the limit onto the limit. The theorem is proved.

Similarly to Theorem 2, with the aid of Lemma 7 one proves

Theorem 3. *If, in the hypotheses of Theorem 2, the bicom pactum X_1 is moreover an open image of D^τ , then the inverse limit of the spectrum $\{X_\alpha; \pi_\beta^\alpha\}$ is an open image of the generalized Cantor discontinuum.*

Theorem 4. *If, in the hypotheses of Theorem 2, for every α the mapping $\pi_\alpha^{\alpha+1}$ exists and is, moreover, a mapping with perfect point-preimages, then the limit of the spectrum $\{X_\alpha; \pi_\beta^\alpha\}$ is homeomorphic to the topological product X_1 by D^τ .*

It is clear that, if the hypotheses of Theorem 4 are satisfied and X_1 is a zero-dimensional compactum or the generalized Cantor discontinuum, then the limit of the spectrum $\{X_\alpha; \pi_\beta^\alpha\}$ is homeomorphic to D^τ .

Corollary 1. *If X is the limit of a transfinite spectrum of zero-dimensional compacta with open mappings, then X is an open image of D^τ .*

Theorem 5. *If X is a dyadic bicom pactum of weight not exceeding $\aleph_1$, then the bicom pactum $\exp X$ is dyadic.*

Proof. As is easy to see, for the proof of the theorem it is enough to show that $\exp D^{\aleph_1}$ is homeomorphic to $D^{\aleph_1}$. Indeed, by virtue of the dyadicity of X there exists a continuous mapping of $D^{\aleph_1}$ onto X . Then, by Lemma 1, there exists a continuous mapping of $\exp D^{\aleph_1}$ onto $\exp X$, and if $\exp D^{\aleph_1}$ is a dyadic bicom pactum, then the theorem will be proved. Let $\{\alpha\}$ be the set of transfinite

numbers less than ω_1 . To each α assign a copy C_α of the Cantor discontinuum and put $X_\alpha = \prod_{\beta < \alpha} C_\beta$. Denoting by π_β^α the natural projection of the product onto a subproduct, we have $\pi_\gamma^\alpha \pi_\beta^\gamma = \pi_\beta^\alpha$, and then $\{X_\alpha; \pi_\beta^\alpha\}$ is an inverse transfinite spectrum whose limit, obviously, is $D^{\aleph_1}$. But the space of closed subsets of a metric compactum is again a metric compactum, and, by Theorem 1, $\exp D^{\aleph_1}$ is the inverse limit of a transfinite spectrum of compacta with mappings open by Lemma 1. Moreover, it is not hard to see that the exponential extension of the mapping π_β^α is a mapping with perfect point-preimages, and, by Theorem 4, in view of Corollary 1, there follows a homeomorphism of $\exp D^{\aleph_1}$ and $D^{\aleph_1}$. The theorem is proved.

With the application of Lemma 1 one also proves

Corollary 2. *If X is an open image of $D^{\aleph_1}$, then $\exp X$ is an open image of $D^{\aleph_1}$.*

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