

ON $\backslash(G\backslash)$ -COMPOSABILITY AND $\backslash(G\backslash)$ - COMPLEMENTABILITY

V. B. Zylev

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.95939>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.3

MATHEMATICS

V. B. Zylev

ON G -COMPOSABILITY AND G -COMPLEMENTABILITY

(Presented by Academician P. S. Novikov on 10 V 1967)

In this note only compact sets that are closures of open sets will be considered; for brevity we shall call them **bodies**.

Let a group G act in a topological space R , possessing the following property: if A and B are bodies, then in the group G there exists a transformation g such that $gA \subset B$.

If the intersections $P_i \cap P_j$, $a_i P_i \cap a_j P_j$ ($i \neq j$), where $a_i \in G$, are either empty or consist of points that are boundary points for the intersecting bodies, and

$$C = \bigcup_{i=1}^n P_i, \quad C' = \bigcup_{i=1}^n a_i P_i,$$

then C and C' will be called **G -composable**, and we shall write

$$C \sim_g C',$$

or simply $C \sim C'$.

If for bodies A and A' there can be found such bodies P_i and such transformations $a_i \in G$ ($i = 1, \dots, n$) that the intersections $P_i \cap P_j$, $a_i P_i \cap a_j P_j$ ($i \neq j$), $A \cap P_i$, $A' \cap a_i P_i$ ($i = 1, \dots, n$), are either empty or consist of boundary points, and

$$A \cup \bigcup_{i=1}^n P_i = A' \cup \bigcup_{i=1}^n a_i P_i,$$

then A and A' will be called **G -complementable**.

Theorem 1. Two G -complementable sets A and A' are always G -composable.

Proof. Given

$$A \cup P = A' \cup P',$$

where

$$P = \bigcup_{i=1}^n P_i, \quad P' = \bigcup_{i=1}^n a_i P_i, \quad a_i \in G \quad (i = 1, \dots, n).$$

If $A = A'$, there is nothing to prove; hence one may assume that one of them is not a part of the other; let, for example, $A' \not\subset A$, then $\overline{A' \setminus A}$ is a body.

By the property of the group G formulated above, there exists a transformation $\alpha \in G$ such that $\alpha P \subset A$, and such an $\alpha' \in G$ that

$$\alpha' P' \subset \overline{A' \setminus A}.$$

Perform this transformation α on P , and remove those parts of A which have been covered by the body αP ; denote them by P^* , and send them by the transformation α^{-1} to the place where the body P was previously located.

We have:

$$(\overline{A \setminus \alpha P}) \cup \alpha^{-1} P^* \sim A,$$

since the body

$$(\overline{A \setminus \alpha P}) \cup \alpha^{-1} P^*$$

is obtained from A by rearranging parts. In the same way we perform the transformation α' on P' , and remove those parts of A' which have become covered by the body $\alpha' P'$; denote them by P'^* , and by the transformation $(\alpha')^{-1}$ send them to the places freed by the removal of P' .

We have:

$$(\overline{A' \setminus \alpha' P'}) \cup (\alpha')^{-1} P'^* \sim A',$$

since the body

$$(\overline{A' \setminus \alpha' P'}) \cup (\alpha')^{-1} P'^*$$

is obtained from A' by rearranging parts.

Introduce the new bodies

$$Q = A \cup P = A' \cup P'$$

and

$$R = \overline{Q \setminus \alpha P \setminus \alpha' P'}.$$

Since $\alpha P \cap \alpha' P'$ is empty, the bodies

$$R \cup \alpha P$$

and

$$R \cup \alpha' P'$$

will be G -composable.

We have the equalities

$$(\overline{A \setminus \alpha P}) \cup \alpha^{-1} P^* = R \cup \alpha' P', \quad (\overline{A' \setminus \alpha' P'}) \cup$$

$$\frac{\cup (\alpha')^{-1} P'^*}{(\overline{A' \setminus \alpha' P'}) \cup (\alpha')^{-1} P'^*} = \overline{R} \cup \alpha P,$$

but earlier we had $(\overline{A \setminus aP}) \cup a^{-1}P^* \sim A$, $(A' \setminus a'P') \cup (a')^{-1}P^* \sim A'$, and, by the transitivity of G -composability, $A \sim A'$. The theorem is proved.

Let us derive from this theorem the following:

Theorem 2. *In three-dimensional Euclidean space, any two polyhedra can be decomposed into a finite number of pairwise similar polyhedra.*

Consider two equal tetrahedra T and T' . It is known that the tetrahedron T can be decomposed into two similar-to- T tetrahedra T_1 and T_2 , and into two prisms Π_1 and Π_2 . The tetrahedron T' can likewise be decomposed into three tetrahedra similar to it and into four prisms: $T' \sim T'_1 \cup T'_2 \cup T'_3 \cup \Pi'_1 \cup \Pi'_2 \cup \Pi'_3 \cup \Pi'_4$. a) $T \sim T_1 \cup T_2 \cup \Pi_1 \cup \Pi_2$. But it is known that there exists a prism Π'_5 such that $\Pi'_5 \sim \Pi'_1 \cup \Pi'_2 \cup \Pi'_3 \cup \Pi'_4$; then, evidently: b) $T' \sim T'_1 \cup T'_2 \cup T'_3 \cup \Pi'_5$.

Applying our theorem to the equalities a) and b) (taking into account that $T = T'$, T_1 is similar to T'_1 , T_2 is similar to T'_2 , and Π and Π'_5 can be decomposed into pairwise similar ones), we obtain

$$\overline{T \setminus T_1 \setminus T_2 \setminus \Pi_1} \sim \Pi_2, \quad \overline{T' \setminus T'_1 \setminus T'_2 \setminus \Pi'_5} \sim T'_3.$$

The prism Π_2 and the tetrahedron T'_3 are similarly composed by virtue of our theorem. Thus, any tetrahedron is similarly composed with a prism. It follows from this that any two three-dimensional polyhedra can be decomposed into a finite number of pairwise similar polyhedra.

In conclusion I express my gratitude to V. A. Efremovich for his help.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
30 III 1967

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.