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Abstract

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MATHEMATICS

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ESTIMATING THE DEFORMATION OF A CONVEX SURFACE AS A FUNCTION OF THE CHANGE IN ITS INTRINSIC METRIC

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Among the theorems on the unique determination of convex surfaces by their intrinsic metric, the greatest attention of geometers has been attracted by the theorem on the unique determination of closed convex surfaces. (For a brief survey and history of the question, see ⁽¹⁾.) The final result here was obtained by A. V. Pogorelov ⁽²⁾. A more general problem of estimating the change of spatial distances between points of a closed convex surface as a function of the change of intrinsic distances was posed by S. E. Cohn-Vossen ⁽³⁾. More precisely, the problem is formulated as follows:

Let F^0 and F^1 be closed convex surfaces, φ a homeomorphism from F^0 onto F^1 , and

$$\varepsilon = \max_{X, Y \in F^0} |\rho_0(X, Y) - \rho_1[\varphi(X), \varphi(Y)]|, \quad (1)$$

where ρ_0 and ρ_1 are, respectively, the distances on the surfaces F^0 and F^1 . It is required to estimate, in terms of ε , the quantity

$$\delta = \max_{X, Y \in F^0} |r(X, Y) - r[\varphi(X), \varphi(Y)]|, \quad (2)$$

where r denotes spatial distances between the points under consideration.

In the present paper it is proved that

$$\delta \leq C\varepsilon^\alpha, \quad (3)$$

where C is a constant depending only on the diameter of the surface F^0 (or F^1), and $\alpha = 1/18$. The best (i.e., largest) value α_0 of the exponent α for which estimate (3) is still valid is unknown. Simple examples show that $\alpha_0 \leq 1/2$.

If the main idea of the proof given below is retained, estimate (3) cannot be obtained with $\alpha > 1/3$.

The connection between the external form of a closed convex surface and its intrinsic metric is expressed analytically (in the case of a regular metric) in the fact that the spatial coordinates of points of the surface satisfy a certain equation of Monge-Ampère type (the Darboux equation), whose coefficients are determined by the intrinsic metric. Direct use of this connection to obtain the estimate we need offers little hope of success. This is due to the fact that small changes of the intrinsic metric may greatly change the coefficients of the Darboux equation mentioned above (the coefficients of the equation contain the second derivatives of the coefficients of the first quadratic form of the surface), and therefore the application of estimates, known for equations of this type, to the solutions does not give an estimate of δ in terms of ε . Such an approach was used in ^(4,5).

The proof of estimate (3) outlined in the present note is based on the following considerations. First of all, it is sufficient to establish an estimate of the form (3) for polyhedra, since after that it is transferred to general convex surfaces by an obvious limiting passage.

We now define the main auxiliary object participating in our proof: an abstract polyhedron.

An ordinary convex polyhedron can be imagined as decomposed into tetrahedra in the following way: take some vertex A_0 of the polyhedron, triangulate all faces not incident with the vertex A_0 , and consider the pyramids whose bases are the triangles of the triangulation and whose common vertex is the point A_0 .

Let us imagine that we have a development S of positive curvature (see ⁽⁶⁾, Ch. 1, § 6), homeomorphic to a sphere and consisting of triangles. Suppose that some vertex A_0 of the development S is singled out, and suppose that to each vertex A_k , different from A_0 , there is assigned a positive number r_k , in such a way that: 1) to the vertices A_k adjacent to A_0 there are assigned the lengths of the edges A_0A_k of the development S ; 2) on any triangle $A_iA_jA_k$ ($i, j, k \neq 0$) of the development S , one can construct, as on a base, a pyramid with lengths of the lateral edges r_i, r_j, r_k , respectively. Such a set of pyramids, together with the rule for gluing them along the lateral faces which is naturally induced by gluing the bases of these pyramids into the development S , we shall call an **abstract polyhedron P with development S and radii r_k** . In the obvious way one defines the interior and boundary edges of the abstract polyhedron and the total dihedral angles at these or those edges. If θ_k is the total angle at an interior edge, then $\omega_k = 2\pi - \theta_k$ will be called the **curvature of the polyhedron P at this edge**. An abstract polyhedron is called **convex** if its total dihedral angle at any boundary edge does not exceed π .

To each abstract polyhedron P we assign a number $H(P)$, which we shall call the **total curvature** of P , and define as follows:

$$H(P) = \sum_i \omega_i r_i + \sum_k (\pi - \alpha_k) l_k, \quad (4)$$

where the first sum is taken over all interior edges, r_i are the lengths of these edges, and ω_i the curvatures at them; the second sum ranges over all boundary edges, where α_k are the total dihedral angles and l_k the lengths of the edges.

For an ordinary Euclidean polyhedron P , all $\omega_i = 0$, and $H(P)$ reduces to twice the integral mean curvature of the surface of this polyhedron.

The total curvature of a convex abstract polyhedron has the following remarkable properties, which allow, in particular, the estimate of δ in terms of ε that we need.

I. Let P^0 be an ordinary convex polyhedron and S^0 its development, and let P^1 be an abstract convex polyhedron with development S^1 , isometric to S^0 ; then

$$H(P^0) \geq H(P^1); \quad (5)$$

moreover, if among the radii r_i^1 of the polyhedron P^1 there are numbers larger than the corresponding (under the isometry) radii r_i^0 of the vertices of the polyhedron P^0 , then

$$H(P^0) - H(P^1) \geq C \lambda_0 \left[\max_i (r_i^1 - r_i^0) \right]^3, \quad (5)$$

where λ_0 is the radius of the largest circle contained in the spherical image of the distinguished vertex A_0 of the polyhedron P^0 , and C is a constant depending only on the intrinsic diameter of the development S^0 .

Thus, in passing from an ordinary polyhedron to an abstract one with an isometric development, the total curvature decreases; moreover, this decrease, in the case where at least one of the radii is lengthened, is sufficiently "large," i.e., it admits estimate (6).

II. On the other hand, whatever the ordinary convex polyhedron P^0 with development S^0 and distinguished vertex A_0 , and whatever the homeomorphic sphere, the development of positive curvature S^1 , and homeomor-

homeomorphism φ of S^0 onto S^1 , one can construct a convex abstract polyhedron P^{10} (a mixed polyhedron) such that: a) its unfolding S^{10} is isometric to S^1 ; b) the radii of the corresponding points of the unfoldings S^0 and S^{10} in the polyhedra P^0 and P^{10} differ by no more than $C^* \varepsilon$; c) the decrease of the total curvature in passing from P^0 to P^{10} , i.e. $H(P^0) - H(P^{10})$, does not exceed $C \varepsilon^{1/3}$, where ε is determined according to (1), and C^* and C are constants.

We shall now derive, relying mainly on properties I and II, the estimate (3) that we need.

Let P^0 and P^1 be ordinary closed convex polyhedra; let S^0 and S^1 be their unfoldings; let φ be a homeomorphism of S^0 onto S^1 , and let ε and δ be the previously introduced (see (1) and (2)) characteristics of the mutual deviation of the intrinsic metrics of S^0 and S^1 and of the spatial dimensions of P^0 and P^1 .

We shall assume that the numbering of the polyhedra P^0 and P^1 themselves and of their vertices A_i^0 and A_i^1 has been chosen so that

$$r(A_0^1, A_1^1) - r(A_0^0, A_1^0) = \max_{i,k} |r(A_i^1 A_k^1) - r(A_i^0 A_k^0)| = \delta.$$

We shall deform the polyhedron P^0 , moving the vertex A_0^0 in the direction of one of the outer normals of the polyhedron P^0 at the point A_0^0 until the spherical image of this vertex contains a certain circle of radius λ , and denote the resulting polyhedron by P_λ^0 , and any of its unfoldings by S_λ^0 .

Denote by P^{01} and P^{10} the mixed polyhedra constructed, the first from the polyhedron P^1 and the unfolding S^0 , the second correspondingly from P^0 and S^1 , and by P_λ^{01} the mixed polyhedron constructed from P^1 and S_λ^0 .

Take the identity

$$\begin{aligned} (H^0 - H_\lambda^0) + (H_\lambda^0 - H_\lambda^{01}) + (H_\lambda^{01} - H^{01}) + (H^1 - H^{10}) \\ = (H^0 - H^{10}) + (H^1 - H^{01}), \end{aligned} \quad (7)$$

which relates the total curvatures of the polyhedra $P^0, P_\lambda^0, P_\lambda^{01}, P^{01}, P^1, P^{10}$, and estimate the left-hand side from below through $\delta, \lambda, \varepsilon$, and the right-hand side from above through ε .

By the construction of P_λ^0 and by property I applied to P_λ^0 and P_λ^{01} , and to P^1 and P^{10} , we have

$$H^0 - H_\lambda^0 \geq -C_1 \lambda^2; \quad H_\lambda^0 - H_\lambda^{01} \geq C_2 \lambda (\delta - C_3 \lambda - C_4 \varepsilon)^3; \quad H^1 - H^{10} \geq 0.$$

The simple nature of the deformation of S^0 into S_λ^0 makes it possible to establish that

$$H_\lambda^{01} - H^{01} \geq 0.$$

On the other hand, by property II applied to P^0 and P^{10} , and to P^1 and P^{01} ,

$$H^0 - H^{10} \leq C_5 \varepsilon^{1/3}; \quad H^1 - H^{01} \leq C_5 \varepsilon^{1/3}.$$

Substitution of all these estimates into (7) gives

$$-C_1\lambda^2 + C_2\lambda(\delta - C_3\lambda - C_4\varepsilon)^3 \leq 2C_5\varepsilon^{1/3}.$$

From this estimate of δ in terms of λ and ε one obtains the best (i.e. with the largest exponent) estimate in terms of ε by taking $\lambda \approx \varepsilon^{1/6}$. In this case it turns out that

$$\delta \leq C\varepsilon^{1/18}.$$

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