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A. B. Vasil'eva, V. A. Tupchiev

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Abstract

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MATHEMATICS

A. B. VASIL' EVA, V. A. TUPCHIEV

ON PERIODIC SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVES, CLOSE TO DISCONTINUOUS ONES

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1. In paper (¹), periodic solutions of systems of ordinary differential equations with a small parameter μ multiplying the derivatives were considered, which, as $\mu \rightarrow 0$, tend to a certain discontinuous periodic function. In the present note we shall investigate periodic solutions of systems of the same type that possess an analogous property. However, the nature of these solutions is entirely different. The asymptotics of such solutions with respect to the parameter μ is also constructed differently.

2. Consider the system of two equations

$$\mu dx/dt = X(x, y), \quad \mu dy/dt = Y(x, y). \quad (1)$$

Suppose that in the phase plane (x, y) the system possesses a cell consisting of saddles $\bar{A}$ and $\bar{B}$, connected by separatrices 1 and 2 and filled with closed trajectories surrounding the equilibrium point O (see Fig. 1), where $ACBDA$ is one of such closed trajectories (the lines $\bar{A}\bar{B}$ and $\bar{C}\bar{D}$ will be needed and will be explained below). An example of a cell of this kind is given by the system $dx/dt = -y + y^3 + xy$, $dy/dt = x$.

Take, among the closed trajectories, a solution of period T . Obviously, if one fixes the phase trajectory and lets μ tend to zero, then the rates of change of x and y will increase and the period corresponding to this trajectory will decrease. If, on the contrary, one fixes the period, then as $\mu \rightarrow 0$ the trajectory will shift and approach the closed curve formed by the separatrices. At the same time the distribution of velocities along the phase trajectory becomes increasingly nonuniform. In regions close to $\bar{A}$ and $\bar{B}$, these velocities are small, since although μ is small, X and Y are also small. Far from $\bar{A}$ and $\bar{B}$, however, the velocities are very large. Thus, in the limit one may expect the appearance of a discontinuous periodic process (in the plane (t, x) or (t, y)). The aim of the

Fig. 1

Figure 1: Fig. 1

present paper is the following: for a given period, to give an asymptotic formula for the laws of motion $x(t, \mu)$ and $y(t, \mu)$ for small μ .

Fig. 1

3. Formula for the period. We shall assume that X and Y possess continuous derivatives up to the second order inclusive in some domain G containing the cell. We shall demonstrate the method in the case when $\bar{A}$ and $\bar{B}$ are connected by the line $Y = 0$, passing through O . In addition, let the line $X = 0$ also pass through O , intersecting the cycle formed by the separatrices at the points $\bar{C}$ and $\bar{D}$ (Fig. 1). The closed trajectory under consideration intersects the lines $Y = 0$ and $X = 0$ at the points A, B and C, D , respectively. We shall specify the line $Y = 0$ parametrically by the arc length s

(let, for definiteness, s be measured from O), the trajectory $ABCD A$ by the arc length l (increasing from A to C), and the curve $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$, formed by separatrices, by the arc length $\bar{l}$ (increasing from $\bar{A}$ to $\bar{C}$). We shall denote the values of the parameters s and l corresponding to all the indicated points by the indices of the points below; for example, the point A corresponds to the parameters s_A and l_A , etc.

The quantity s_A may serve as a parameter of the family of closed trajectories; the period of a closed trajectory is a function of s_A : $T(s_A)$. Let us pose the following problem: find an asymptotic formula for the period $T(s_A)$ in the case when $\Delta s_A = s_{\bar{A}} - s_A$ is small, i.e., the trajectory is close to the cycle formed by separatrices.

We emphasize from the very beginning that the arrangement of the lines $X = 0$ and $Y = 0$ adopted here is not essential. The method is quite applicable also to other cases, say, from $\bar{A}$ the line $Y = 0$ may go inside the cell, and from $\bar{B}$ the line $X = 0$, which intersect at O (this will be the case, for example, for the system $dx/dt = y^2 - 1$, $dy/dt = -x^2 + 1$), etc.

By $\bar{X}$ and $\bar{Y}$ we shall denote the right-hand sides of (1), taken along the separatrix and thus being functions of $\bar{l}$. The values of the right-hand sides of (1) and of other occurring functions of two variables at the point A (and also at other points indicated in Fig. 1) will be denoted by $X(A)$, etc. We also introduce the notation $\lambda_1(x, y)$, $\lambda_2(x, y)$, and $\Delta(x, y)$, respectively, for the characteristic roots and the determinant of the matrix composed of the partial derivatives of X and Y ,

$$\begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix}.$$

Let, at the points $\bar{A}$ and $\bar{B}$, λ_1 correspond to separatrix 1 and λ_2 to separatrix 2, and let $\lambda_1(\bar{A}) > 0$ (then $\lambda_2(\bar{A}) < 0$, $\lambda_1(\bar{B}) < 0$, and $\lambda_2(\bar{B}) > 0$). Denote by T_{AC}, T_{CB}, T_{BD} , and T_{DA} the times during which the representative point passes, respectively, along the segments AC, CB, BD , and DA . The notation T_{DAC} and T_{CBD} used in Sec. 5 has an analogous meaning.

Theorem 1. *Under the assumptions made in Sec. 3, for T_{AC} there is the asymptotic formula*

$$T_{AC} = -\frac{\mu}{\lambda_1(\bar{A})} \ln \Delta s_A + \mu I_{\bar{A}\bar{C}} + \mu \omega(\Delta s_A), \quad (2)$$

where

$$I_{\bar{A}\bar{C}} = \frac{1}{\lambda_1(\bar{A})} \ln |\alpha(\bar{A})(\bar{l}_{\bar{C}} - \bar{l}_{\bar{A}})| + \int_{\bar{l}_{\bar{A}}}^{\bar{l}_{\bar{C}}} \left\{ \frac{1}{\sqrt{\bar{X}^2 + \bar{Y}^2}} - \frac{1}{\lambda_1(\bar{A})(\bar{l} - \bar{l}_{\bar{A}})} \right\} d\bar{l}, \quad (3)$$

$$\alpha(\bar{A}) = \frac{\lambda_2(\bar{A}) - \lambda_1(\bar{A})}{\lambda_2(\bar{A})} \frac{\sqrt{Y_y(\bar{A})^2 + Y_x(\bar{A})^2}}{\sqrt{[Y_y(\bar{A}) - \lambda_1(\bar{A})]^2 + Y_x(\bar{A})^2}}$$

and $\omega(\Delta s_A) \rightarrow 0$ as $\Delta s_A \rightarrow 0$.*

Analogous formulas hold for $T_{AD} = -T_{DA}$, $T_{BC} = -T_{CB}$, and T_{BD} . T_{AD} is obtained from (2) by replacing $\lambda_1(\bar{A})$ by $\lambda_2(\bar{A})$ and $\bar{C}$ by $\bar{D}$; T_{BC} is obtained from (2) by replacing $\bar{A}$ by $\bar{B}$ and A by B ; T_{BD} by the same replacement from T_{AD} . In what follows we shall also use the expressions I_{CB}, I_{DA} , which, by definition, are equal to $I_{CB} = -I_{BC}$, $I_{DA} = -I_{AD}$.

The proof of Theorem 1 can be carried out by introducing, in a neighborhood of $\bar{A}$, new variables ξ, η , in which system (1) has, in the linear approximation—

* Here and in what follows, $\omega(\alpha)$ denotes a quantity infinitely small as $\alpha \rightarrow 0$; moreover, we agree to denote by one and the same letter ω quantities of different infinitesimal order, if their order is not of essential significance.

diagonal form, and using certain properties of trajectories in a neighborhood of a saddle, described, for example, in (2), § 30.

4. The connection formula is $\Delta s_A = \bar{s}_A - s_A$ and $\Delta s_B = \bar{s}_B - s_B$. To answer the question posed in item 2, one must specify Δs_A as a function of μ in such a way that the period is equal to the prescribed value T . But since T_{AC} and T_{DA} are expressed in terms of Δs_A , and T_{CB} and T_{BD} in terms of Δs_B , it is necessary first to establish a relation between Δs_A and Δs_B .

Denote by Z the vector with coordinates X, Y . Suppose, in addition to the requirements of item 3, that $Y_x Y_y \neq 0$ at the points $\bar{A}$ and $\bar{B}$.

Theorem 2. *Under the assumptions made in items 3, 4, the following formula holds*

$$[R_1(\bar{B})]^\gamma (1 + \omega(\Delta s_B)) \Delta s_B = [R_1(\bar{A})]^\gamma (1 + \omega(\Delta s_A)) \Delta s_A, \quad (4)$$

where

$$\begin{aligned} R_1(\bar{A}) &= \frac{Y_x(\bar{A}) \Delta(\bar{A})}{Y_x^2(\bar{A}) + Y_y^2(\bar{A})} \exp [K_{\overline{AC}} + \operatorname{div} Z(\bar{A}) I_{\overline{AC}}], \\ \gamma &= \frac{\lambda_1(\bar{A}) Y_y}{\lambda_1(\bar{A}) - \lambda_2(\bar{A})} = \frac{\lambda_1(\bar{B})}{\lambda_1(\bar{B}) - \lambda_2(\bar{B})}, \\ K_{\overline{AC}} &= \int_{\bar{A}}^{\bar{C}} \frac{\operatorname{div} Z - \operatorname{div} Z(\bar{A})}{\sqrt{X^2 + Y^2}} d\bar{l}; \end{aligned} \quad (5)$$

$I_{\overline{AC}}$ is defined by formula (3); $R_1(\bar{B})$ is obtained from $R_1(\bar{A})$ by replacing $\bar{A}$ by $\bar{B}$.

The proof of Theorem 2 is based on the known connection formula for Δs_A and Δs_B (see, for example, (3), Ch. 2, § 3).

5. Finding the asymptotics in μ for the times T_{DAC} and T_{CBD} of traversing the portions of the trajectory DAC and CBD for a prescribed period T . Asymptotics of the laws of motion $x(t, \mu)$ and $y(t, \mu)$. From (2) and the corresponding expressions for T_{AD}, T_{BC}, T_{BD} , we obtain a representation of

$$T = T_{DA} + T_{AC} + T_{CB} + T_{BD}$$

in terms of $\Delta s_A, \Delta s_B$, and μ , as well as analogous representations for

$$T_{DAC} = T_{DA} + T_{AC}$$

and

$$T_{CBD} = T_{CB} + T_{BD}.$$

Then, expressing Δs_B in terms of Δs_A from (4), for prescribed T we obtain an equation relating Δs_A and μ . From this equation one obtains an asymptotic representation of Δs_A in terms of μ ; moreover, it is interesting to note that $\Delta s_A(\mu)$ turns out to be exponentially small. Substituting $\Delta s_A(\mu)$ into the expressions for T_{DAC} and T_{CBD} , we obtain the desired asymptotic representation of these quantities in μ .

Theorem 3. *For the times T_{DAC} and T_{CBD} of traversing the portions DAC and CBD , for prescribed T , the following asymptotic formulas in the parameter μ hold:*

$$T_{DAC} = T_{DAC}^{(0)} + \mu T_{DAC}^{(1)} + \mu \omega(\mu) = \frac{1}{\lambda_2(\bar{B}) - \lambda_2(\bar{A})} [\lambda_2(\bar{B}) T + \mu \kappa] + \mu \omega(\eta), \quad (6)$$

$$T_{CBD} = T_{CBD}^{(0)} + \mu T_{CBD}^{(1)} + \mu\omega(\mu) = \frac{1}{\lambda_2(\bar{B}) - \lambda_2(\bar{A})} [-\lambda_2(\bar{A})T - \mu\kappa] + \mu\omega(\mu).$$

Here

$$I_{DAC} = I_{DA} + I_{AC}, \quad I_{CBD} = I_{CB} + I_{BD}, \quad \kappa = \ln R_1(\bar{A})/R_1(\bar{B}) - \lambda_2(\bar{A})I_{DAC} + \lambda_2(\bar{B})I_{CBD}.$$

Thus, in the zero approximation, T_{DAC} and T_{CBD} are parts of the period T , proportional to $\lambda_2(\bar{B})$ and $-\lambda_2(\bar{A})$, respectively, i.e.

$$T_{DAC}^{(0)} + T_{CBD}^{(0)} = T, \quad T_{DAC}^{(0)}/T_{CBD}^{(0)} = -\lambda_2(\bar{B})/\lambda_2(\bar{A}),$$

and the corrections of order μ have the same absolute magnitude and opposite signs, i.e.

$$T_{DAC}^{(1)} = -T_{CBD}^{(1)}.$$

In order to construct the asymptotics of the law of motion $z(t, \mu)$ (let z denote x and y together), it is necessary to fix the origin for measuring t .

Let, for definiteness, the point D correspond to $t = 0$. Then for $z(t, \mu)$ we have the boundary-value problem

$$z(0, \mu) = z_D = z_{\bar{D}} + \omega(\Delta s_A) = z_{\bar{D}} + \omega(\mu),$$

$$z(t_0, \mu) = z_C = z_{\bar{C}} + \omega(\Delta s_A) = z_{\bar{C}} + \omega(\mu).$$

Here $t_0 = T_{DAC}$. In the spirit of the notation used above, l_C , etc., z_C should be understood as the collection of the coordinates x_C, y_C of the point C ; the same also applies to z with indices $D, \bar{C}, \bar{D}$. This boundary-value problem differs from the problem considered in (4) only in that the boundary value of the unknown function is given with accuracy $\omega(\mu)$, and the position of the endpoint t_0 with accuracy $\mu\omega(\mu)$. This does not prevent the application of the asymptotic scheme constructed in (4), and for $z(t, \mu)$ one can write an asymptotic formula similar to those given in (4).

Theorem 4. For $z(t, \mu)$ the following asymptotic representation holds:

$$z(t, \mu) = z_{\bar{A}} + \Pi_0 z + \Pi_1 z + \omega(\mu), \quad (7)$$

where $\Pi_0 z$ and $\Pi_1 z$ are boundary-layer functions determined by the systems ($i = 1, 2$)

$$\mu \frac{d\Pi_i z}{dt} = Z(x_{\bar{A}} + \Pi_i x, y_{\bar{A}} + \Pi_i y)$$

and by the initial conditions at the points $t = 0$ and $t = \tilde{t}_0 = T_{DAC}^{(0)} + \mu T_{DAC}^{(1)}$, respectively,

$$\Pi_0 z|_{t=0} = z_{\bar{D}} - z_{\bar{A}}, \quad \Pi_1 z|_{t=\tilde{t}_0} = z_{\bar{C}} - z_{\bar{A}},$$

and $\omega(\mu)$ tends uniformly to zero as $\mu \rightarrow 0$ for $0 \leq t \leq \hat{t}_0$.

The $\Pi_i z$, thus, are constructed from segments of separatrices. We point out that, since at their initial points the slope of the Π -functions is of order $1/\mu$, the inclusion of the term $\mu T_{DAC}^{(1)}$ in the asymptotics of t_0 is necessary to ensure asymptotic accuracy $\omega(\mu)$ in formula (7).

Analogously, $z(t, \mu)$ is constructed on the interval $t_0 \leq t \leq T$.

From (7) it is clear that, as $\mu \rightarrow 0$, $z(t, \mu)$ tends to the discontinuous function $\bar{z}$

$$\bar{z} = \begin{cases} z_{\bar{A}}, & 0 < t < \frac{\lambda_2(\bar{B})}{\lambda_2(\bar{B}) - \lambda_2(\bar{A})} T, \\ z_{\bar{B}}, & \frac{\lambda_2(\bar{B})}{\lambda_2(\bar{B}) - \lambda_2(\bar{A})} T < t < T. \end{cases}$$

In conclusion, we note that analogous phenomena are also possible for nonautonomous cases, and they can likewise be investigated by the methods of work (4).

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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