

ONE VARIATIONAL PROBLEM IN PARAMETRIC FORM

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Abstract

Full Text

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MATHEMATICS

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ONE VARIATIONAL PROBLEM IN PARAMETRIC FORM

Let $H(x_1, x_2, x_3)$ be a positive homogeneous function of the first degree that is strictly convex. This means that for any $\lambda \geq 0$

$$H(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda H(x_1, x_2, x_3)$$

and the second differential d^2H is a positive definite form.

We consider the following variational problem. Find a regular surface $F : r = r(u, v)$ with a given spherical image ω , with a given support function along the boundary $(rn = \psi)$, which gives the functional

$$I(F) = \iint_F H(A_1, A_2, A_3) du dv$$

a stationary value ($\delta I = 0$). Here A_1, A_2, A_3 are the components of the vector product $r_u \times r_v$. By a regular surface we mean the image of an arbitrary domain of the uv -plane by means of a regular vector function $r(u, v)$. The usual requirement $r_u \times r_v \neq 0$ is not assumed, since geometric singularities on the surface are not excluded.

Theorem. *Whatever the function H of class $C^{2,\lambda}$, the domain ω of class $C^{2,\lambda}$ lying entirely on one hemisphere of the unit sphere, and the function ψ , prescribed on the boundary γ of the domain ω , the variational problem stated above is always solvable in the class of surfaces $C^{2,\lambda}$.*

Proof. We shall seek the solution of our variational problem among surfaces F with nonpositive Gaussian curvature and one-to-one spherical image. Such a surface can be specified by its support function $\Phi(n) = rn$, where r is the vector of a point of the surface and n is the unit normal vector at this point.

By the condition of the theorem, the domain ω (the spherical image of the sought surface) lies entirely on one hemisphere. Without loss of generality, one may assume that this hemisphere is

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_3 > 0.$$

The domain ω is projected from the origin of coordinates onto a certain domain $\bar{\omega}$ in the plane $x_3 = 1$, bounded by a curve $\bar{\gamma}$ of class $C^{2,\lambda}$. To each point n of the domain ω under this projection there corresponds a certain point (x_1, x_2) of the domain $\bar{\omega}$ in the plane $x_3 = 1$. In our assumption the numbers x_1, x_2 may be regarded as coordinates on the surface F .

By virtue of the homogeneity of the function H , we have

$$H(A_1, A_2, A_3) du dv = H(n_1, n_2, n_3) |r_u \times r_v| du dv = H(n_1, n_2, n_3) d\sigma,$$

where n_1, n_2, n_3 are the components of the vector n (the unit normal to the surface F), and $d\sigma$ is the area element of the surface F .

Put $\Phi(x) = (rx)$, where x is any nonzero vector directed into the domain ω , and r is the vector of a point of the surface F at which the normal (n) has the direction of the vector x . The function $\Phi(x)$ is positively homogeneous of the first degree. The coordinates of points of the surface F are expressed through the derivatives of the function Φ . Namely,

$$x'_1 = \partial\Phi/\partial x_1, \quad x'_2 = \partial\Phi/\partial x_2, \quad x'_3 = \partial\Phi/\partial x_3.$$

Setting $\Phi(x_1, x_2, 1) = \varphi(x_1, x_2)$, we obtain

$$x'_1 = \varphi_1, \quad x'_2 = \varphi_2, \quad x'_3 = \varphi - x_1\varphi_1 - x_2\varphi_2,$$

where φ_1 and φ_2 are the first derivatives of the function φ with respect to the variables x_1 and x_2 , respectively. With the aid of these formulas for the coordinates of points of the surface F , we find the expression for the area element

$$d\sigma = -(\varphi_{11}\varphi_{22} - \varphi_{12}^2)\sqrt{1 + x_1^2 + x_2^2},$$

where φ_{ij} are the second derivatives of the function φ .

Noting that

$$n_1 = \frac{x_1}{\sqrt{1 + x_1^2 + x_2^2}}, \quad n_2 = \frac{x_2}{\sqrt{1 + x_1^2 + x_2^2}}, \quad n_3 = \frac{1}{\sqrt{1 + x_1^2 + x_2^2}},$$

we obtain

$$I(F) = - \iint_{\bar{\omega}} h(x_1, x_2) (\varphi_{11}\varphi_{22} - \varphi_{12}^2) dx_1 dx_2,$$

where, for brevity, we have denoted $H(x_1, x_2, 1) \equiv h(x_1, x_2)$.

Let us form the Euler equation for the functional I . We obtain

$$h_{22}\varphi_{11} - 2h_{12}\varphi_{12} + h_{11}\varphi_{22} = 0. \quad (*)$$

From the positive definiteness of the form d^2H follows the positive definiteness of the form d^2h and, consequently, the ellipticity of the equation for the function φ .

The finding of the surface F solving our variational problem is reduced to solving equation $(*)$ in the domain $\bar{\omega}$ with the boundary condition

$$\varphi|_{\gamma} = \psi\sqrt{1 + x_1^2 + x_2^2}.$$

As is known, this boundary-value problem is always solvable in the class $C^{2,\lambda}$, if the coefficients of the equation (h_{ij}) belong to $C^{0,\lambda}$, and the curve γ bounding the domain is of class $C^{2,\lambda}$ ⁽¹⁾. In our case these conditions are satisfied.

If $\varphi(x_1, x_2)$ is a solution of the boundary-value problem for equation $(*)$, then the surface F solving our variational problem is given by the equations

$$x'_1 = \partial\varphi/\partial x_1, \quad x'_2 = \partial\varphi/\partial x_2, \quad x'_3 = \varphi - x_1\partial\varphi/\partial x_1 - x_2\partial\varphi/\partial x_2.$$

We note that the curvature of the surface F has the sign of the discriminant $\varphi_{11}\varphi_{22} - \varphi_{12}^2$ and, consequently, by virtue of the ellipticity of equation $(*)$, is nonpositive, as was assumed. The theorem is proved.

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CITED LITERATURE

¹ K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.

Note: Figure translations are in progress. See original paper for figures.

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