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Abstract

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MATHEMATICS

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APPROXIMATE DIFFERENTIATION BY MEANS OF LAGRANGE AND HERMITE INTERPOLATION POLYNOMIALS

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As is well known, one of the methods for the approximate computation of a derivative is to replace it by the differentiated Lagrange interpolation polynomial (¹⁻³). In connection with this, for a function $f(x)$ continuously differentiable on the segment $[-1, 1]$, consider the quantity

$$R'_n(f, x, M) = f'(x) - L'_n(f, x, M), \quad (1)$$

where

$$L_n(f, x, M) = \sum_0^n f(x_k^{(n+1)}) l_{k,n+1}(x), \quad (2)$$

$$l_{k,n+1}(x) = \frac{\omega_{n+1}(x)}{\omega'_{n+1}(x_k^{(n+1)})(x - x_k^{(n+1)})}, \quad \omega_{n+1}(x) = \prod_0^n (x - x_k^{(n+1)}), \quad (3)$$

and $M = \|x_k^{(n+1)}\|$ is a certain infinite triangular matrix of nodes located on the segment $[-1, 1]$. In papers (^{1, 4, 6}) cases are considered in which the quantity (1) is estimated in terms of the derivative of order n of the function $f(x)$.

In the present paper estimates are given for the rate of decrease of the quantity (1) under the condition that the function $f(x)$ is only continuously differentiable on $[-1, 1]$, and the matrix of nodes is defined by one of the formulas

$$x_0^{(n+2)} = 1, \quad x_{n+1}^{(n+2)} = -1, \quad x_k^{(n+2)} = \cos \frac{2k-1}{2n} \pi \quad (k = 1, \dots, n); \quad (4)$$

$$x_k^{(n+1)} = \cos k \frac{\pi}{n} \quad (k = 0, \dots, n). \quad (5)$$

From (3) and (4) it follows that $\omega_{n+2}(x) = (x^2 - 1)\tilde{T}_n(x)$, where $\tilde{T}_n(x)$ is the Chebyshev polynomial of the first kind with leading coefficient equal to one. Similarly, under condition (5) we have $\omega_{n+1}(x) = (x^2 - 1)\tilde{U}_{n-1}(x)$, where $\tilde{U}_{n-1}(x)$ is the Chebyshev polynomial of the second kind.

Let us denote the matrices of nodes (4) and (5), respectively, by M_1 and M_2 , and put

$$D_n(x, M_1) = \sum_1^n |l'_{k,n+2}(x)| \sin \theta_k^{(n+2)}, \quad (6)$$

$$D_n(M_1) = \max_{x \in [-1, 1]} D_n(x, M_1). \quad (7)$$

Lemma 1. If the matrix of nodes is defined by formula (4), then the inequalities hold

$$\frac{2}{\pi} n \ln \frac{1}{\sin \pi/n} < D_n(M_1) < \frac{26}{\pi} n \ln n + O(n),$$

$$c_1 n \leq \max_{x \in \Delta_n} D_n(x, M_1) \leq c_2 n,$$

where the set Δ_n consists of the points $x_k^{(n)} = \cos k\pi/n$ ($k = 1, \dots, n-1$) and the zeros of the derivative of the polynomial $\omega_{n+2}(x) = (x^2 - 1)\tilde{T}_n(x)$.

The proofs of these inequalities are obtained with the aid of formulas (2) and (3), if one takes into account that, by virtue of (4), here we have $\omega_{n+2}(x) = (x^2 - 1)\pi_n(x)$. Analogous results also hold for the matrix M_2 .

Theorem 1. If the function $f(x)$ is continuously differentiable on the segment $[-1, 1]$ and $\omega(\delta, f')$ is the modulus of continuity of its derivative $f'(x)$, then for the Lagrange interpolation polynomial corresponding to the matrix M_1 the inequalities

$$|R'_{n+1}(f, x, M_1)| \leq c_3 \omega(1/n, f') \ln n, \quad x \in [-1, 1], \quad (8)$$

$$|R'_{n+1}(f, x, M_1)| \leq c_4 \omega(1/n, f'), \quad x \in \Delta_n \quad (9)$$

hold.

Under the conditions of Theorem 1, in paper ⁽⁷⁾ the existence is proved of polynomials $\{Q_n(x)\}$ for which, for $k = 0, 1$, the estimates

$$|f^{(k)}(x) - Q_n^{(k)}(x)| \leq c_5 \frac{\sqrt{1-x^2}^{1-k}}{n^{1-k}} \omega\left(\frac{\sqrt{1-x^2}}{n}, f'\right), \quad x \in [-1, 1].$$

Using these polynomials in the inequality

$$|f'(x) - L'_{n+1}(f, x, M_1)| \leq |f'(x) - Q'_n(x)| + \sum_1^n |f(x_k^{(n+2)}) - Q_n(x_k^{(n+2)})| |l'_{k,n+2}(x)|,$$

with the aid of Lemma 1 we obtain the estimates (8) and (9).

It is not hard to prove that estimate (9) is sharp in order. Indeed, if, for example, $f_1(x) = -x^2(1-x^2)$ for $-1 \leq x < 0$ and $f_1(x) = x^2(1-x^2)$ for $0 \leq x \leq 1$, then we shall have $f'_1(x) \in \text{Lip } 1$ and $f'_1(0) = 0$. Therefore, for $n = 2p$ we find

$$L'_{n+1}(f_1, 0, M_1) = -\frac{1}{n \cos \pi/2n}.$$

If the function $f(x)$ has p continuous derivatives on the segment $[-1, 1]$, with the p -th derivative having modulus of continuity $\omega(\delta, f^{(p)})$, then in the case of the matrix M_1 the estimates

$$|R'_{n+1}(f, x, M_1)| \leq c_6 \frac{\ln n}{n^{p-1}} \omega\left(\frac{1}{n}, f^{(p)}\right), \quad x \in [-1, 1], \quad (10)$$

$$|R'_{n+1}(f, x, M_1)| \leq c_7 \frac{1}{n^{p-1}} \omega\left(\frac{1}{n}, f^{(p)}\right), \quad x \in \Delta_n \quad (11)$$

are valid.

Analogous facts also hold for the matrix M_2 . Further, one may consider a matrix M for which the conditions

$$0 < c_8 \leq \max_{[x_k^{(n)}, x_{k+1}^{(n)}]} |\omega_n(x)| / \max_{[x_i^{(n)}, x_{i+1}^{(n)}]} |\omega_n(x)| \leq c_9 < \infty,$$

are fulfilled, where $\omega_n(x)$ is defined by formula (3) for the matrix M , and $x_0^{(n)} = 1$, $x_{n+1}^{(n)} = -1$. The convergence of the interpolation process for such matrices is considered in papers (8,9). It is not hard to show that estimates of the form (8), (10) also hold here.

Let us now consider the quantity

$$r_{2n+1}(f, x, M) = f'(x) - H'_{2n+1}(f, x, M), \quad (12)$$

where

$$H_{2n+1}(f, x, M) = \sum_0^n f(x_k^{(n+1)}) U_{k,n+1}(x) + \sum_0^n f'(x_k^{(n+1)}) V_{k,n+1}(x), \quad (13)$$

$$U_{k,n+1}(x) = \left[1 - \frac{\omega''_{n+1}(x_k^{(n+1)})}{\omega'_{n+1}(x_k^{(n+1)})} (x - x_k^{(n+1)}) \right] l_{k,n+1}^2(x), \quad (14)$$

$$V_{k,n+1}(x) = (x - x_k^{(n+1)}) l_{k,n+1}^2(x). \quad (15)$$

Similarly to (6) and (7), here we set

$$B_{n+1}(f, x, \alpha) = \sum_0^n |f'_k(x)| \sin^\alpha \theta_k^{(n+1)}, \quad \alpha \geq 0, \quad (16)$$

$$B_{n+1}(f, E, \alpha) = \max_{x \in E} B_{n+1}(f, x, \alpha), \quad E = [-1, 1]. \quad (17)$$

Lemma 2. If the matrix of nodes is determined by formula (4), then for $x \in [-1, 1]$ the inequality

$$B_{n+1}(V, x, 1) \leq \frac{2}{n} |\omega'_{n+2}(x)| \left\{ \frac{4}{\pi} |\tilde{T}'_n(x)| \ln n + O(1) \right\} + O(1).$$

holds.

Lemma 3. If the matrix of nodes is M_1 , then the inequalities

$$c_{10} n \ln n \leq B_{n+1}(U, [-1, 1], 2) \leq c_{11} n \ln n,$$

$$c_{12} n \leq B_{n+1}(U, \Delta_n) \leq c_{13} n.$$

hold.

Theorem 2. If the matrix of nodes is M_1 and the function $f(x)$ is twice continuously differentiable on the segment $[-1, 1]$, then the estimates

$$|r'_{2n+3}(f, x, M_1)| \leq c_{14} \frac{\ln n}{n} \omega\left(\frac{1}{n}, f''\right), \quad x \in [-1, 1], \quad (18)$$

$$|r'_{2n+3}(f, x, M_1)| \leq c_{15} \frac{1}{n} \omega\left(\frac{1}{n}, f''\right), \quad x \in \Delta_n. \quad (19)$$

hold.

Theorem 3. If the matrix of nodes is M_2 and the function $f(x)$ is continuously differentiable on the segment $[-1, 1]$, then the inequalities

$$|r'_{2n+1}(f, x, M_2)| \leq c_{16} \omega\left(\frac{1}{n}, f'\right) \ln n, \quad x \in [-1, 1], \quad (20)$$

$$|r'_{2n+1}(f, x, M_2)| \leq c_{17} \omega\left(\frac{1}{n}, f'\right), \quad x \in \Delta'_n, \quad (21)$$

hold, where Δ'_n is the set of zeros of the derivative of the polynomial $\omega_{n+1}(x) = (x^2 - 1)U_{n-1}(x)$.

As examples show, inequalities (19) and (21) are exact in order. If the function $f(x)$ has p continuous derivatives, then the estimates (18)–(21) are modified analogously to (10)–(11).

One can consider analogous questions for the matrices M_1^* and M_2^* , obtained respectively from the matrices M_1 and M_2 by deleting the end points $x = \pm 1$.

Theorem 4. If the matrix of nodes is M_1^* and the function $f(x)$ is twice continuously differentiable on the segment $[-1, 1]$, then the estimates

$$|R'_{n-1}(f, x, M_1^*)| \leq c_{18} \omega\left(\frac{1}{n}, f''\right) \ln n, \quad x \in [-1, 1],$$

$$|R'_{n-1}(f, x, M_1^*)| \leq c_{19} \frac{1}{n} \omega\left(\frac{1}{n}, f''\right), \quad x \in \Delta''_n,$$

hold, where Δ''_n is the set of zeros of the derivative of the polynomial $\omega_n(x) = \tilde{T}_n(x)$.

Theorem 5. If the matrix of nodes is M_1^* and the function $f(x)$ is continuously differentiable on the segment $[-1, 1]$, then the estimates

$$|R'_{n-1}(f, x, M_1^*)| \leq c_{20} \frac{1}{\sqrt{1-x^2}} \omega\left(\frac{1}{n}, f'\right) \ln n, \quad x \in (-1, 1)$$

$$|R'_{n-1}(f, x, M_1^*)| \leq c_{21} \omega\left(\frac{1}{n}, f'\right), \quad x \in \Delta''_n.$$

Estimates of the same form are also valid for the case of Hermite interpolation. For example, if $f'(x) \in \text{Lip } 1$ and $x \in \Delta''_n$, then the estimate

$$|r'_{2n-1}(f, x, M_1)| \leq c_{22}/n, \quad x \in \Delta''_n$$

holds.

On the other hand, if one sets $f_2(x) = -x^2/2$ for $-1 \leq x < 0$ and $f_2(x) = x^2/2$ for $0 \leq x \leq 1$, then $f'_2(x) \in \text{Lip } 1$, $f'_2(0) = 0$, and for $n = 2p$ we have

$$H'_{2n-1}(f_2, 0, M_1^*) = -\frac{1}{2} \frac{1}{n^2 \sin \pi/2n}.$$

In conclusion, we note some results on the approximate differentiation of periodic functions. Denote by m_1^* the matrix of nodes

$$x_k^{(n)} = \frac{2n - 2k + 1}{2n} \pi \quad (k = 1, \dots, 2n),$$

and let there correspond to it the trigonometric interpolation polynomial $t_{n-1}(f, x, m_1^*)$.

Theorem 6. If an even periodic function $f(x)$ is continuously differentiable p times, and $E_n(f^{(p)})$ is the best approximation of $f^{(p)}(x)$ by trigonometric polynomials of degree not exceeding n , then the estimate

$$\left| f^{(p)}(x) - t_{n-1}^{(p)}(f, x, m_1^*) \right| \leq (c_{23} + c_{24} \ln n) E_n(f^{(p)}).$$

holds.

In particular, if $p = 1$, then the first estimate of Theorem 5 follows from this. Analogous results are obtained also in the case where the matrices M_1 and M_2 are transformed in the corresponding way.

In all the estimates given above, certain (not best possible) values of the constants have been computed.

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