

# BANACH COMMUTATIVE SYMMETRIC ALGEBRAS OF OPERATORS IN THE PONTRYAGIN SPACE $\backslash(\backslash\mathbb{P}_1\backslash)$

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**Abstract**

**Full Text**

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*MATHEMATICS*

A. I. LOGINOV

# **BANACH COMMUTATIVE SYMMETRIC ALGEBRAS OF OPERATORS IN THE PONTYAGIN SPACE $\Pi_1$**

*(Presented by Academician I. M. Vinogradov, 15 VI 1967)*

Let  $R$  be a commutative symmetric algebra of bounded operators in the Pontryagin space  $\Pi_1$ . In the works of M. A. Naimark <sup>(1,2)</sup> a classification of such algebras was given. It was shown that, under the assumptions of separability of  $R$  and  $\Pi_1$ , all such algebras (with the exception of certain simple cases) are determined, up to equivalence, by means of a certain model. There the conditions for equivalence of algebras defined by such a model were also found. In fact, two distinct types of inequivalent models were indicated: a) and b) (see <sup>(1)</sup>); the algebras corresponding to them will be called, respectively, **singular** and **nonsingular**.

The operators of a singular algebra  $R$  are given by the formulas

$$A\xi_0 = \lambda(A)\xi_0,$$

$$Ap(t) = (p, p_{A^*})\xi_0 + A_1p = \tag{1}$$

$$= - \int_{T_1} (A_1(t) - \lambda(A))(p(t), \xi(t)) d\sigma \cdot \xi_0 + A_1(t)p(t),$$

$$Aq = (q, q_{A^*})\xi_0 + \lambda(A)q,$$

$$A\eta_0 = \gamma(A)\xi_0 + \lambda(A)\eta_0 + p_A + q_A =$$

$$= \gamma(A)\xi_0 + \lambda(A)\eta_0 + (A_2(t) - \lambda(A))\xi(t) + q_A,$$

where  $T$  is a bicomact space,  $\sigma$  is a regular Borel measure on  $T$ , and  $\lambda(A) = A_1(t_0)$  and  $T_1 = T - \{t_0\}$ ,  $t_0 \in T$ . The point  $t_0 \in T$  will be called a **singular point** of the singular algebra  $R$ .

For the operators of a nonsingular algebra  $R$  we have formulas of the form

$$\begin{aligned} A\xi_0 &= \lambda(A)\xi_0, \\ Ap(t) &= (p, p_{A^*})\xi_0 + A_1p = \\ &= - \int_T (A_1(t) - \lambda(A))(p(t), \xi(t)) d\sigma \cdot \xi_0 + A_1(t)p(t), \end{aligned} \quad (2)$$

$$\begin{aligned} A\eta_0 &= \gamma(A)\xi_0 + \lambda(A)\eta_0 + p_A = \\ &= \gamma(A)\xi_0 + \lambda(A)\eta_0 + (A_1(t) - \lambda(A))\xi(t), \end{aligned}$$

with now  $\lambda(A) \neq A_1(t_0)$  for any point  $t_0 \in T$ . (For a detailed explanation of these formulas and of the notation connected with them, see <sup>(2)</sup>.)

Thus, each operator  $A \in R$  is defined by the system of parameters  $\{A_1(t), q_A, \gamma(A), \lambda(A)\}$ . When  $A$  runs through the algebra  $R$ ,  $A_1(t)$  runs through a certain symmetric algebra of functions  $R_1 \subset C(T)$ , uniformly dense in  $C(T)$ ; the vectors  $q_A$  form a certain linear manifold  $\mathcal{L}$ , and  $\gamma(A)$  and  $\lambda(A)$  are linear functionals on  $R$  (see <sup>(2)</sup>). The totality  $\mathcal{B}$  of all systems  $\{A_1(t), q_A, \gamma(A), \lambda(A)\}$  corresponding to a certain algebra  $R$  forms a linear manifold in the totality  $G$  of all systems  $\{A_1(t), q_A, \gamma(A), \lambda(A)\}$ , when all parameters independently run through their ranges of values. This manifold is called the **defining manifold** of the algebra  $R$ . The problem

of the present work is to describe the determining manifolds of complete singular and nonsingular algebras with identity.

1. Denote

$$M_0 = \{A : A \in R, \lambda(A) = 0\}.$$

Then, since  $\lambda(A) = \overline{\lambda(A^*)}$ ,  $M_0$  is a symmetric maximal ideal of the algebra  $R$ . Moreover,  $R = 1 + M_0$ , since for  $B = A - \lambda(A) \cdot 1$  we have  $\lambda(B) = 0$  and  $B \in M_0$ .

Denote  $M_1 = \{A : A \in R, A_1(t) \equiv 0\}$ . Then, since the homomorphism  $A \rightarrow A_1(t)$  is continuous and symmetric (see <sup>(2)</sup>),  $M_1$  is a closed symmetric ideal in  $R$ .

Denote also by  $\widetilde{M}_0$  the image of the ideal  $M_0$  under the homomorphism  $A \rightarrow A_1(t)$ .  $\widetilde{M}_0$  is a symmetric maximal ideal in  $R_1$ , proper or improper. The following assertions are valid with respect to  $\widetilde{M}_0$ .

- I. If  $R$  is nonsingular, then the ideal  $\widetilde{M}_0$  is uniformly dense in  $C(T)$ .
- II. If  $R$  is singular, then the ideal  $\widetilde{M}_0$  is uniformly dense in  $C_{t_0}(T)$ .
2. Every operator of the algebra given by formulas (1) or (2) is bounded. The operator norm in the algebra can be expressed in terms of the parameters of  $R$  via the kernel of the homomorphism  $R \rightarrow R_1$  isomorphic to the algebra  $R_1$ , i.e. the operator norm up to equivalence has the form

$$\begin{aligned} \|A\| &= |A_1| + |\gamma(A)| + |p_A| + |q_A| = \sup_{t \in T} |A_1(t)| + |\gamma(A)| + |q_A| + \\ &+ \left[ \int_{T_1} |A_1(t) - A_1(t_0)|^2 d\mu \right]^{1/2}, \end{aligned} \quad (3)$$

where  $d\mu = (\xi(t), \xi(t)) d\sigma$ .

For a nonsingular algebra we have

$$\begin{aligned} \|A\| &= |\lambda(A)| + |A_1| + |\gamma(A)| = \\ &= |\lambda(A)| + |\gamma(A)| + \sup_{t \in T} |A_1(t)|. \end{aligned} \quad (4)$$

3. All the preceding assertions are valid in the case of an arbitrary algebra, complete or incomplete. We now impose the condition of completeness on the algebra  $R$ . Since the algebra  $R_1$  is homomorphic to the algebra  $R$ , the factor algebra of  $R$  by the kernel of the homomorphism  $R \rightarrow R_1$  is isomorphic to the algebra  $R_1$ , i.e. the algebra  $R/M_1$  is algebraically isomorphic to the algebra  $R_1$ . Since the homomorphism  $R \rightarrow R_1$  is continuous and, consequently, the ideal  $M_1$  is closed, by a known theorem (see, for example, (3), Ch. I, § 4.3) and the completeness of  $R$  in the norm  $\|A\|$ , the algebra  $R/M_1$  and, consequently, the isomorphic algebra  $R_1$ , will be complete in the natural norm in the quotient space  $R/M_1$ . If  $\widetilde{A}$  is the class of elements from  $R$  congruent modulo  $M_1$  and  $A_1(t) \in R_1$  is the function corresponding to it, then this natural norm is equal to

$$\|\widetilde{A}\| = \|A_1\| = \|A_1(t)\| = \inf_{A \in \widetilde{A}} \|A\|.$$

From formulas (3) and (4) it follows that this norm has the form

$$\|\tilde{A}\| = \sup_{t \in T} |A_1(t)| + \inf_{A \in \tilde{A}} (|\lambda(A)| + |\gamma(A)|) \quad (5)$$

for a nonsingular algebra  $R$ , and

$$\begin{aligned} \|\tilde{A}\| = & \sup_{t \in T} |A_1(t)| + \\ & + \inf_{A \in \tilde{A}} \left[ |\lambda(A)| + |\gamma(A)| + |q_A| + \left( \int_{T_1} |A_1(t) - A_1(t_0)|^2 d\mu \right)^{1/2} \right] \end{aligned} \quad (6)$$

for a singular algebra  $R$ .

Thus, in the case of completeness of  $R$  in its norm, the algebra  $R_1$  is not only uniformly dense in  $C(T)$ , but also complete in the norm (5) or (6). This fact makes it possible to describe concretely the structure of the algebra  $R_1$  and of the determining manifolds for complete algebras.

4. Applying the arguments of Sec. 3 and taking assertion I into account, we obtain that, in the case of a complete nonsingular algebra,  $R_1 = C(T)$ , and for the defining manifold the following holds.

**Theorem 1.** *Every nonsingular complete algebra  $R$  is equivalent to an algebra given by the formulas:*

$$A\xi_0 = \lambda\xi_0,$$

$$Ap(t) = A_1(t)p(t),$$

$$A\eta_0 = \gamma\xi_0 + \lambda\eta_0,$$

where  $\lambda$  and  $\gamma$  are arbitrary complex numbers, and  $A_1(t)$  is an arbitrary function from  $C(T)$ .

5. In the study of singular algebras it is useful to consider the following collection of functions from  $R_1$ . Denote by  $S$  the linear span of all possible products of functions from  $R_1$ .  $S$  is a symmetric ideal in  $R_1$ . For  $S$  the following holds.

**Lemma.** *The ideal  $S$  is dense, in the norm*

$$\sup_{t \in T} |A_1(t)| + \left( \int_{T_1} |A_1(t)|^2 d\mu \right)^{1/2}$$

in the ring of all continuous functions on  $T$  that vanish at 0 at the point  $t_0$ , for which this norm is finite.

A detailed consideration of the defining manifolds of complete singular algebras, using assertion II, the arguments of Sec. 3, and the lemma, leads to the fact that in this case  $R_1$  consists of all continuous functions on  $T$  for which the norm is finite:

$$\|A_1\| = \sup_{t \in T} |A_1(t)| + \left( \int_{T_1} |A_1(t) - A_1(t_0)|^2 d\mu \right)^{1/2}. \quad (7)$$

For defining manifolds of complete singular algebras the following holds.

**Theorem 2.** *Every complete singular algebra is given by a defining manifold of the form*

$$\{A_1(t), q_A, \gamma, \lambda(A) = A_1(t_0)\},$$

where  $\gamma$ ,  $q_A$ , and  $A_1(t)$  independently run through, respectively: the set of complex numbers, a certain complete space  $\mathcal{E}$ , and the ring of all functions for which the norm (7) is finite.

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*Note: Figure translations are in progress. See original paper for figures.*

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