

ON THE METHOD OF TOTAL TRANSPORTS OF STOKMAN– SVERDRUP–MUNK

A. I. Felzenbaum

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Abstract

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GEOPHYSICS

A. I. FEL' ZENBAUM

ON THE METHOD OF TOTAL TRANSPORTS OF STOKMAN–SVERDRUP–MUNK

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In studying a steady current in the ocean, we shall proceed from the following system of equations and boundary conditions:

$$A_l \Delta u + A_z \frac{\partial^2 u}{\partial z^2} + \Omega v = -g \frac{\partial \zeta}{\partial x} + \frac{g}{\rho_0} \int_0^z \frac{\partial \rho}{\partial x} dz, \quad (1)$$

$$A_l \Delta v + A_z \frac{\partial^2 v}{\partial z^2} - \Omega u = -g \frac{\partial \zeta}{\partial y} + \frac{g}{\rho_0} \int_0^z \frac{\partial \rho}{\partial y} dz;$$

$$S_x = -\partial \psi / \partial y, \quad S_y = \partial \psi / \partial x; \quad (2)$$

$$w = \int_z^H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz; \quad (3)$$

$$u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z} \left(\chi_1 \frac{\partial \rho}{\partial z} \right) + \chi_2 \Delta \rho; \quad (4)$$

$$\text{at } z = 0 \quad A_z \partial u / \partial z = -T_x / \rho_0, \quad A_z \partial v / \partial z = -T_y / \rho_0; \quad (5)$$

$$a_0 \partial \rho / \partial z + b_0 \rho = \Gamma_0; \quad (6)$$

$$\text{at } z = H$$

$$u = v = 0; \quad (7)$$

$$a_H \partial \rho / \partial z + b_H \rho = \Gamma_H; \quad (8)$$

on l

$$u = v = 0; \quad (9)$$

$$a_l \partial \rho / \partial n + b_l \rho = \Gamma_l. \quad (10)$$

In (1)–(10), u, v, w are the components of the current velocity; g is the acceleration of gravity; ρ is density, $\rho_0 = \text{const}$ is the mean density; ζ is the level; H is the depth of the basin; l is the shore of the closed basin (vertical wall) and n is the direction of the normal to it; A_z, A_l are the coefficients of vertical and horizontal exchange of momentum; χ_1, χ_2 are the coefficients of vertical and horizontal turbulent diffusion; T_x, T_y are the components of the tangential wind stress; Δ is the Laplace operator; $a_0, b_0, \Gamma_0, \dots$ are known quantities; S_x, S_y are the components of the total transport, ψ is the integral stream function.

The problem under consideration can be solved, taking into account the bottom relief and the peculiarities introduced by the equator, numerically [1], if condition (9) is adopted. With such an approach, the terms due to horizontal exchange will enter the right-hand side of the equation for the function ψ through S_x^* and S_y^* . Here, restricting ourselves to the case $H = \text{const}$, we shall proceed in another way. For $H = \text{const}$, from (1) and (5) we obtain

$$\begin{aligned} A_l \Delta S_x + \frac{T_x}{\rho_0} - \frac{R_x}{\rho_0} + \Omega S_y &= -gH \frac{\partial \zeta}{\partial x} + \frac{g}{\rho_0} \frac{\partial Q}{\partial x}, \\ A_l \Delta S_y + \frac{T_y}{\rho_0} - \frac{R_y}{\rho_0} - \Omega S_x &= -gH \frac{\partial \zeta}{\partial y} + \frac{g}{\rho_0} \frac{\partial Q}{\partial y}, \end{aligned} \quad (11)$$

where the notation has been introduced

$$Q = \int_0^H \int_0^z \rho \, d\eta \, dz, \quad R_x = \rho_0 A_z \left(\frac{\partial u}{\partial z} \right)_{z=H}, \quad R_y = \rho_0 A_z \left(\frac{\partial v}{\partial z} \right)_{z=H}. \quad (12)$$

Cross-differentiation of (11) gives $\beta = d\Omega/dy$

$$A_l \Delta \Delta \psi - \beta \frac{\partial \psi}{\partial x} - \frac{1}{\rho_0} \text{rot}_z \bar{R} = -\frac{1}{\rho_0} \text{rot}_z \bar{T}. \quad (13)$$

If, instead of (7), the condition of slip without friction is imposed at the bottom,

$$\text{at } z = H \quad R_x = R_y = 0, \quad (7')$$

then, instead of (13), we obtain*

$$A_l \Delta \Delta \psi - \beta \frac{\partial \psi}{\partial x} = -\frac{1}{\rho_0} \text{rot}_z \bar{T}. \quad (14)$$

This equation, with the boundary conditions

$$\text{on } l \quad \psi = \partial \psi / \partial n = 0, \quad (15)$$

which follow from (9), uniquely determines the field of total transports as a function of the wind field and independently of the processes described by equation (4) and conditions (6), (8), (10). From this incorrect result** it follows that, for a baroclinic ocean, condition (7') cannot be adopted. Thus, if equation (14) of the Stommel-Sverdrup-Munk theory is valid, it is so only for a homogeneous ocean. This, however, is also doubtful, since its solution depends neither on the position of the basin relative to the equator nor on whether the basin under consideration is shallow or deep***. Apparently, if equation (14) is satisfied at all, it is only in the particular case $\rho = \text{const}$ and of a deep basin located at a sufficient distance from the equator.

Let us now return to our problem, restricting ourselves to the case of a deep basin with a horizontal bottom, situated outside the equatorial zone of the ocean.

Represent the total transport $\bar{S}$ as the sum of the purely drift total transport $\bar{S}_1$, caused directly by the dragging action of the wind, the total transport $\bar{S}_2$, caused by the analogous effect of the bottom, and the total transport $\bar{S}_3 = \bar{S} - \bar{S}_1 - \bar{S}_2$. In determining $\bar{S}_1$ and $\bar{S}_2$ we shall neglect horizontal exchange, and in determining $\bar{S}_3$, vertical exchange. Then from (1), (5), (7), (12) we obtain

$$S_{x1} = T_y / \Omega \rho_0, \quad S_{y1} = T_x / \Omega \rho_0; \quad (16)$$

$$S_{x2} = -R_y / \Omega \rho_0 = -U / 2a, \quad S_{y2} = R_x / \Omega \rho_0 = -V / 2a; \quad (17)$$

$$S_{x3} = \frac{gH}{\Omega} \frac{\partial \xi}{\partial y} + \frac{A_l}{\Omega} \Delta S_y - \frac{g}{\Omega \rho_0} \frac{\partial Q}{\partial y}, \quad S_{y3} = -\frac{gH}{\Omega} \frac{\partial \xi}{\partial x} - \frac{A_l}{\Omega} \Delta S_x + \frac{g}{\Omega \rho_0} \frac{\partial Q}{\partial x}, \quad (18)$$

where a is the Ekman parameter; $D = \pi/a$ is the friction depth;

$$U = (u)_{z=H-D} + (v)_{z=H-D}; \quad V = (v)_{z=H-D} - (u)_{z=H-D}.$$

Since the influence of turbulent friction between horizontal layers of seawater is practically manifested only in the surface and near-bottom friction layers, for a deep sea ($2aH \gg 1$) from (1) with completely

* Equation (14) was obtained for $\beta = 0$ by V. B. Shtokman ⁽²⁾, for $A_l = 0$ by G. Sverdrup ⁽³⁾, and in the general case by V. Munk ⁽⁴⁾.

** In particular, from (14) and (15) we arrive at the incorrect conclusion that there is no integral circulation in the absence of wind.

*** In reality, both of these factors introduce a number of features into the distribution of total transports ⁽⁵⁻⁷⁾.

with sufficient accuracy

$$(u)_{z=H-D} = \frac{g}{\Omega} \frac{\partial \xi}{\partial y} - \frac{g}{\Omega \rho_0} \int_0^{H-D} \frac{\partial \rho}{\partial y} dz + \frac{A_l}{\Omega} \Delta(v)_{z=H-D}, \quad (19)$$

$$(v)_{z=H-D} = -\frac{g}{\Omega} \frac{\partial \xi}{\partial x} + \frac{g}{\Omega \rho_0} \int_0^{H-D} \frac{\partial \rho}{\partial x} dz - \frac{A_l}{\Omega} \Delta(u)_{z=H-D}.$$

Summing (16), (17), (18) and taking (19) into account, we obtain

$$S_x = \frac{T_y}{\Omega \rho_0} + \frac{g}{2\Omega a} \frac{\partial \xi}{\partial x} + \frac{g}{2\Omega a} (2aH-1) \frac{\partial \xi}{\partial y} + \frac{A_l}{\Omega} \Delta S_y - \frac{g}{\Omega \rho_0} \frac{\partial Q}{\partial y} + \frac{g}{2\Omega \rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial x} \right) dz - \frac{A_l}{2\Omega a} \Delta V, \quad (20)$$

$$S_y = -\frac{T_x}{\Omega \rho_0} - \frac{g}{2\Omega a} (2aH-1) \frac{\partial \xi}{\partial x} + \frac{g}{2\Omega a} \frac{\partial \xi}{\partial y} - \frac{A_l}{\Omega} \Delta S_x + \frac{g}{\Omega \rho_0} \frac{\partial Q}{\partial x} - \frac{g}{2\Omega \rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial x} \right) dz - \frac{A_l}{2\Omega a} \Delta U,$$

whence

$$\frac{\partial \xi}{\partial x} = \frac{2a}{g} \frac{1}{1 + (2aH-1)^2} \left\{ \Omega S_x - \frac{T_y}{\rho_0} - A_l \Delta S_y + \frac{g}{\rho_0} \frac{\partial Q}{\partial y} - \frac{g}{2\rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial x} \right) dz + \frac{A_l}{2a} \Delta V - (2aH-1) \right\}$$

$$\frac{\partial \xi}{\partial y} = \frac{2a}{g} \frac{1}{1 + (2aH-1)^2} \left\{ \Omega S_y + \frac{T_x}{\rho_0} + A_l \Delta S_x - \frac{g}{\rho_0} \frac{\partial Q}{\partial x} + \frac{g}{2\rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial x} \right) dz - \frac{A_l}{2a} \Delta U + (2aH-1) \right\}$$

Since for a deep basin $2aH \gg 1^*$, it follows from (21), with quite sufficient accuracy, that

$$\frac{\partial \xi}{\partial x} = \frac{1}{gH} \left\{ \frac{\Omega S_x}{2aH} - \Omega S_y - A_l \Delta S_x - \frac{T_x}{\rho_0} + \frac{1}{2aH} \frac{g}{\rho_0} \frac{\partial Q}{\partial y} + \frac{g}{\rho_0} \frac{\partial Q}{\partial x} - \frac{g}{2\rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial x} \right) dz + \frac{1}{2aH} \left[- \right. \right.$$

$$\left. \frac{\partial \xi}{\partial y} = \frac{1}{gH} \left\{ \frac{\Omega S_y}{2aH} + \Omega S_x - A_l \Delta S_y - \frac{T_y}{\rho_0} - \frac{1}{2aH} \frac{g}{\rho_0} \frac{\partial Q}{\partial x} + \frac{g}{\rho_0} \frac{\partial Q}{\partial y} - \frac{g}{2\rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial x} \right) dz + \frac{1}{2aH} \left[A_l \Delta \right. \right. \right.$$

* $2aH = 10^2$ for $\Omega = 10^{-4} \text{ sec}^{-1}$ (latitude about 45°) and $A_z = 10^2 \text{ cm}^2/\text{sec}$, $H = 1 \text{ km}$ or $A_z = 4 \cdot 10^2 \text{ cm}^2/\text{sec}$

$$\left. + \frac{g}{2\rho_0 a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial x} \right) dz + A_l H \Delta V - \frac{A_l}{2a} \Delta U \right] \Bigg\}.$$

Cross-differentiation of (22), taking (2) into account, gives

$$\frac{\Omega}{2H} \left[\frac{\partial}{\partial x} \left(\frac{1}{a} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{a} \frac{\partial \psi}{\partial y} \right) \right] - A_l \Delta \Delta \psi + \beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0} \text{rot}_z T + \frac{g}{2\rho_0 H} \left\{ \frac{\partial}{\partial x} \left[\frac{1}{a} \frac{\partial Q}{\partial x} + \frac{H}{a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial x} \right) dz \right] \right.$$

Equation (23) relates the integral circulation to the wind field and to the distribution of the density of seawater. This equation, together with equation (4), can be solved numerically, and all the desired functions are thereby calculated. Thus, the determination of the total transports remains an intermediate link in the solution of the entire problem (1).

It follows from equation (23) that integral circulation exists even in the absence of wind: $\psi \neq 0$, provided that "nonmechanical" (climatological) factors also act.

It is important to note that even if the integral circulation is produced only by the wind (in the absence of nonmechanical factors), it is not determined by this wind uniquely. In this case the function ψ depends not only on the wind, but also on $\varkappa_1(z)$ and on the constants $a_0, \Gamma_c, b_H, \Gamma_H, a_l$ (for $b_0 = a_H = b_l = \Gamma_l = 0$).

Comparing (13) and (23), we obtain

$$\text{rot}_z \bar{R} = \frac{\Omega \rho_0}{2H} \left[\frac{\partial}{\partial x} \left(\frac{1}{a} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{a} \frac{\partial \psi}{\partial y} \right) \right] - \frac{g}{2H} \left\{ \frac{\partial}{\partial x} \left[\frac{1}{a} \frac{\partial Q}{\partial x} + \frac{H}{a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial x} \right) dz \right] + \frac{\partial}{\partial y} \left[\frac{1}{a} \frac{\partial Q}{\partial y} - \frac{H}{a} \int_0^{H-D} \left(\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial x} \right) dz \right] \right\}$$

from which it is seen that bottom friction depends substantially on the inhomogeneity of seawater and on the depth of the basin.

In the case $\rho = \text{const}$, instead of (23) we shall have

$$\frac{\Omega}{2H} \left[\frac{\partial}{\partial x} \left(\frac{1}{a} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{a} \frac{\partial \psi}{\partial y} \right) \right] - A_i \Delta \Delta \psi + \beta \frac{\partial \psi}{\partial x} = \frac{1}{\rho_0} \text{rot}_z \bar{T}. \quad (25)$$

The first term on the left-hand side of (25) is due to bottom friction, the second to horizontal exchange of momentum. When the depth of the basin or the coefficient of horizontal exchange is decreased (increased), the role of bottom friction increases (decreases). In limiting cases we arrive at equation (8, 5), close to Stommel's equation (9), or to Munk's equation (14).

Marine Hydrophysical Institute
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.

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