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Abstract

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CYBERNETICS AND CONTROL THEORY

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APPROXIMATION OF DYNAMIC CHARACTERISTICS IN THE CLASS OF GENERALIZED FUNCTIONS

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Let $y_o(t)$ and $y_m(t)$ be the values of the outputs of the object and of the model, respectively. For real objects $x(t)$ is a finite function, i.e., one that vanishes outside a certain finite interval $[t_0, t_1]$. Then, as is known ⁽¹⁾, the relation between the output and the input of the object has the form

$$y_o(t) = \begin{cases} 0, & t < t_0, \\ \int_{t_0}^t x(u)k_o(t-u) du, & t_0 \leq t \leq t_1, \\ \int_{t_0}^{t_1} x(u)k_o(t-u) du, & t > t_1. \end{cases} \quad (1)$$

An analogous relation (2) is valid for $y_m(t)$. Applying the direct and inverse Fourier transforms to relations (1) and estimating $|y_o(t) - y_m(t)|$, we obtain

$$|y_o(t) - y_m(t)| = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} [\Phi_o(i\omega) - \Phi_m(i\omega)] X(i\omega) e^{i\omega t} d\omega \right| \leq \frac{\max_{t_0 \leq t \leq t_1} |x(t)|(t_1 - t_0)}{2\pi} \int_{-L}^L |\Phi_o(i\omega) - \Phi_m(i\omega)| d\omega +$$

where $\Phi_o(i\omega)$ and $\Phi_m(i\omega)$ are the transfer functions of the object and the model; $[-L, L]$ is a finite interval on the axis (its choice will be discussed below);

$$\int_{-\infty}^{\infty}$$

is the integral over the remaining part of the axis.

Let us first estimate the integral over the finite part of the axis

$$I_1 = \int_{-L}^L |\Phi_o(i\omega) - \Phi_m(i\omega)| d\omega. \quad (3)$$

The integrand is the error of approximation of the transfer function of the object on the interval $[-L, L]$ by means of the interpolation polynomial ⁽²⁾

$$u_n^+(\Phi, \omega) = \sum_{k=0}^n a_k \left(\frac{\omega + i\alpha}{\omega - i\alpha} \right)^k, \quad a_k = \frac{1}{n+1} \sum_{j=0}^n \Phi(i\omega_j) \left(\frac{\omega_j + i\alpha}{\omega_j - i\alpha} \right)^{-k}, \quad (4)$$

$$\omega_j = \alpha \operatorname{ctg} \frac{\theta_j}{2}, \quad \theta_j = \frac{\pi}{n+1} (2j+1), \quad -\pi \leq \theta_j \leq \pi, \quad \alpha > 0.$$

Make the change of variables

$$\omega = i\alpha \frac{\tau + 1}{\tau - 1} = \alpha \operatorname{ctg} \frac{\theta}{2}, \quad \tau = \frac{\omega + i\alpha}{\omega - i\alpha} = e^{i\theta}, \quad -\pi \leq \theta \leq \pi,$$

and obtain an estimate for the approximation of the function $\varphi(\tau) = \Phi(i\alpha \frac{\tau+1}{\tau-1})$ on the part of the unit circle γ' that does not include a certain neighborhood of the singular point $\tau = 1$.

Lemma 1. The estimate

$$|u_n^+(\varphi, \tau)| \leq \begin{cases} \left\lfloor \frac{4}{\pi} + \frac{2}{\pi} \ln \left(\frac{2}{\pi} (n+1) \right) \right\rfloor \max_{\tau \in \gamma'} |\varphi(\tau)| & \text{for even } n, \\ \left\lfloor \frac{2}{\pi} + 1 + \frac{1}{\pi} \ln \left(\frac{2}{\pi} (n+1) \right) \right\rfloor \max_{\tau \in \gamma'} |\varphi(\tau)| & \text{for odd } n, \end{cases} \quad (5)$$

(6)

holds, where the estimate (5) cannot be improved in the sense of its attainability.

The proof of estimates (5) and (6) follows easily from relation (5.5) of work ⁽³⁾.

Consider an estimate of the accuracy of approximation of the function $\varphi(\tau)$, specified on γ' , by means of partial sums in Faber polynomials. For this purpose we construct the function $\tau = \psi(w)$, which conformally maps the exterior of the unit disk onto the exterior of the arc γ' ⁽⁴⁾. It has the form

$$\tau = \psi(w) = w \frac{iw \cos \alpha/2 - 1}{w + i \cos \alpha/2}, \quad \tau \in \gamma', \quad (7)$$

where α is the angle between the radius drawn to the endpoint of the arc γ' and the positive direction of the abscissa axis; $w = \Phi(\tau)$ is the function inverse to $\psi(w)$.

Let $\chi^+(w)$ be the function analytic inside the unit disk ($|w| \leq 1$), obtained from the function $\chi(w) = \varphi[\psi(w)]$ by the Sokhotskii formula ⁽⁵⁾

$$\chi^+(w) = \frac{1}{2}\chi(w) + \frac{1}{2\pi i} \int_{\gamma} \frac{\chi(u)}{u-w} du, \quad P_n(\tau) = \sum_{k=0}^n a_k \Phi_k(\tau), \quad (8)$$

where $\Phi_k(\tau)$ are the Faber polynomials for the continuum γ' , and the a_k are such that

$$\left| \chi^+(w) - \sum_{k=0}^n a_k w^k \right| = \inf_{a_k} \left| \chi^+(w) - \sum_{k=0}^n a_k w^k \right| = \rho_n[\chi^+(w)].$$

In the notation introduced above, the following theorem holds:

Theorem 1. The estimate

$$\max_{\tau \in \gamma'} \left| \varphi(\tau) - \sum_{k=0}^n a_k \Phi_k(\tau) \right| \leq 3\rho_n[\chi^+(w)] \quad (9)$$

is valid.

Proof. For the Faber polynomials the relation ⁽⁵⁾

$$\Phi_k(\tau) = \frac{w_1^k + w_2^k}{2} + \frac{1}{2\pi i} \int_{L_1} \frac{[\Phi(\zeta)]^k d\zeta}{\zeta - \tau}, \quad \tau \in L_1, \quad \psi(w_1) = \psi(w_2) = \tau, \quad (10)$$

holds, where L_1 is a closed curve consisting of two banks of γ' . Further,

$$\begin{aligned} \varphi(\tau) - \sum_{k=0}^n a_k \Phi_k(\tau) &= \frac{1}{2} \left[\chi(w_1) - \sum_{k=0}^n a_k w_1^k \right] + \frac{1}{2} \left[\chi(w_2) - \sum_{k=0}^n a_k w_2^k \right] \\ &+ \frac{1}{2\pi i} \int_{|w|=1} \frac{\chi(w) - \sum_{k=0}^n a_k w^k}{\psi(w) - \psi(\tau)} \psi'(w) dw. \end{aligned}$$

Using the relation

$$\chi(w) = \chi^+(w) - \chi^-(w)$$

and taking into account that

$$\frac{1}{2\pi i} \int_{|w|=1} \frac{\chi^-(w)\psi'(w)}{\psi(w) - \psi(\tau)} dw = -\frac{1}{2} [\chi^-(w_1) - \chi^-(w_2)],$$

$$\frac{\psi'(w)}{\psi(w) - \psi(\tau)} = \frac{1}{w - \tau} - \frac{1}{w + i \cos \alpha/2} + \frac{1}{w - \tau^*}, \quad \tau^* = \frac{1 - i\tau \cos \alpha/2}{\tau + i \cos \alpha/2},$$

we arrive at the proof of estimate (9).

The obtained estimate (9) for the accuracy of approximation of the function $\varphi(\tau)$, specified on γ' , by means of partial sums of Faber polynomials is analogous to the results given in works (6–9). The method of proving estimate (9) differs from the methods used in works (6–8).

Theorem 2. The estimate

$$\max_{\tau \in \gamma'} |\varphi(\tau) - u_n^+(\varphi, \tau)| \leq \begin{cases} 3 \left[1 + \frac{4}{\pi} + \frac{2}{\pi} \ln \left(\frac{2}{\pi} (n+1) \right) \right] \rho_n[\chi^+(w)] & \text{for } n \text{ even,} \\ 3 \left[2 + \frac{2}{\pi} + \frac{1}{\pi} \ln \left(\frac{2}{\pi} (n+1) \right) \right] \rho_n[\chi^+(w)] & \text{for } n \text{ odd.} \end{cases} \quad (11) \quad (12)$$

The proof of Theorem 2 follows from Theorem 1 and Lemma 1. As a result, we obtain an estimate for I_1 (for example, for n even)

$$I_1 \leq 6L \left[1 + \frac{4}{\pi} + \frac{2}{\pi} \ln \left(\frac{2}{\pi} (n+1) \right) \right] \rho_n[\chi^+(w)]. \quad (13)$$

Let us now estimate the integral over the remaining part of the axis

$$I_2 = \left| \int_{-\infty}^{\infty'} [\Phi_0(i\omega) - \Phi_M(i\omega)] X(i\omega) e^{i\omega t} d\omega \right|. \quad (14)$$

We shall carry out the estimate under the following restrictions on $X(i\omega)$ and $\Phi_0(i\omega)$:

$$1. \quad X(i\omega) = \frac{1}{\omega^{k+1}} \sum_{j=1}^r c_j e^{i\omega \alpha_j} + X_1(i\omega), \quad |X_1(i\omega)| \leq \frac{c}{|\omega|^p},$$

$$2. \quad \Phi_0(i\omega) = \omega^k \sum_{s=1}^l d_s e^{i\omega \beta_s} + \Phi_1(i\omega), \quad |\Phi_1(i\omega)| \leq \bar{c} |\omega^{k-1}|, \quad (15)$$

$$k \geq 0, \quad p > k + 1, \quad c_j \geq 0, \quad d_s \geq 0,$$

c_j, d_s are real, and c and $\bar{c}$ are absolute constants.

Further,

$$I_2 \leq \left| \int_{-\infty}^{\infty'} \Phi_0(i\omega) X(i\omega) e^{i\omega t} d\omega \right| + \left| \int_{-\infty}^{\infty'} \Phi_M(i\omega) X(i\omega) \cos \omega t d\omega \right| + \left| \int_{-\infty}^{\infty'} \Phi_M(i\omega) X(i\omega) \sin \omega t d\omega \right|.$$

Lemma 2. The estimate

$$\left| \int_L^{\infty'} \Phi_M(i\omega) X(i\omega) \cos \omega t d\omega \right| \leq \frac{4}{L^{k+1}} \sum_{j=1}^r c_j \left[\frac{2\pi|A_0|}{t + \alpha_j} + \frac{c(n)n}{k+1} \right] + \frac{2c}{L^{p-1}} \left[\frac{|A_0|}{n-1} + \frac{2c(n)n}{(p+1)L^2} \right], \quad (16)$$

where A_0 is the first coefficient of the expansion of $\Phi_M(i\omega)$ into partial fractions.

$$\Phi_M(i\omega) = \sum_{k=0}^n \alpha_k \left(\frac{i\omega - \alpha}{i\omega + \alpha} \right)^k = A_0 + \frac{A_1}{i\omega + \alpha} + \dots + \frac{A_n}{(i\omega + \alpha)^n}, \quad (17)$$

$$c(n) = \left[\frac{4}{\pi} + \frac{2}{\pi} \ln \left(\frac{2}{\pi} (n+1) \right) \right] \max_j |\varphi(\tau_j)|.$$

Proof. The estimate

$$\left| \int_L^{\infty} \frac{\cos \omega t}{\omega^m} d\omega \right| \leq \frac{2\pi}{L^m t}, \quad m \geq 1. \quad (18)$$

is valid.

From relation (17) we have

$$\Phi_M - A_0 = \frac{A_1}{i\omega + \alpha} + \frac{A_2}{(i\omega + \alpha)^2} + \dots + \frac{A_n}{(i\omega + \alpha)^n} = u_n^+(\varphi, \tau) - u_n^+(\varphi, 1).$$

Further, using Bernstein's inequality, we obtain

$$|u_n^+(\varphi, \tau) - u_n^+(\varphi, 1)| \leq \max_{\tau} |[u_n^+(\varphi, \tau)]'| |\tau - 1| \leq c(n)n \cdot 2/|\omega - i|,$$

$$|\Phi_M(i\omega) - A_0| \leq |A_0| + 2c(n)n/\sqrt{1 + \omega^2}. \quad (19)$$

Using relations (18) and (19), we arrive at the proof of relation (16).

Theorem 3. If relation (15) is valid, then

$$I_2 \leq \frac{2}{L} \left(4\pi \sum_{j=1}^r \sum_{s=1}^l \frac{c_j d_s}{t + \alpha_j + \beta_s} + \bar{c} \sum_{j=1}^r c_j \right) + \frac{2c}{L^{p-k-1}} \left(\frac{\sum_{s=1}^l d_s}{k-p+1} + \frac{\bar{c}}{(k-p)L} \right) + \frac{4}{L^{k+1}} \sum_{j=1}^r c_j \left[\frac{2\pi|A_0|}{t + \alpha_j} + \frac{c(n)n}{k-p+1} \right]$$

The proof follows from relations (16)–(19).

As a result, under assumption (15), taking into account relations (2), (13), and (20), we obtain the following theorem.

Theorem 4. The estimate

$$|y_0(t) - y_M(t)| \leq 6L[1+c(n)]\rho_n[\chi^+(w)] + \frac{2}{L} \left(4\pi \sum_{j=1}^r \sum_{s=1}^l \frac{c_j d_s}{t + \alpha_j + \beta_s} + \bar{c} \sum_{j=1}^r c_j \right) + \frac{2c}{L^{p-k-1}} \left(\frac{\sum_{s=1}^l d_s}{k-p+1} + \frac{\bar{c}}{(k-p)L} \right) + \frac{4}{L^{k+1}} \sum_{j=1}^r c_j \left[\frac{2\pi|A_0|}{t + \alpha_j} + \frac{c(n)n}{k-p+1} \right]$$

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Note: Figure translations are in progress. See original paper for figures.

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