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Abstract

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MATHEMATICS

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HARNACK' S INEQUALITY FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER OF CORDES TYPE

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In this note we shall consider the linear elliptic equation of the second order

$$Lu \equiv \sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} = 0. \quad (1)$$

We shall call such an equation an equation of **Cordes type** in some domain D , if there exists a positive constant $s < n$ such that

$$L(1/|x - x^0|^s) \geq 0, \quad (2)$$

$x \in D$, x^0 being any fixed point of R_n .

Let, for equation (1), the inequality

$$\sum_{i,k=1}^n a_{ik} \xi_i \xi_k \geq \alpha |\xi|^2 \quad (3)$$

be satisfied.

It is easy to verify that if $\sup \sum_{i=1}^n a_{ii} / \alpha < n + 2$, the equation is of Cordes type.

We shall assume that equation (1) is of Cordes type. Nothing more will be assumed about its coefficients, including measurability.

By a solution of the equation we shall mean a classical solution—a continuous function having the derivatives entering into the equation and reducing it to an identity.

For an equation of Cordes type, Harnack's inequality is proved. We shall denote by Q_R the ball of radius R with center at the origin of coordinates.

Lemma 1. Let a domain D be situated in the ball Q_4 , have boundary points on the surface S_4 of the ball Q_4 , and intersect the ball Q_1 . Denote by Γ that part of the boundary of D which lies strictly inside Q_4 . Let H be the intersection of the complement of the domain D with the ball Q_1 . Let an equation (1) of Cordes type be defined in D , and let $u(x)$ be its solution, positive in D , continuous in $\bar{D}$, and vanishing on Γ . Then

$$\max_{\bar{D}} u \geq (1 + \eta \operatorname{mes} H) \max_{\bar{D} \cap Q_1} u,$$

where $\eta > 0$ is a constant depending on the constant s in inequality (2) and on the dimension n of the space.

Proof. Put

$$\int_{Q_1} \frac{dx}{|x|^s} = K. \quad (4)$$

Then for any $y \in R_n$

$$U(y) = \frac{1}{K} \int_H \frac{dx}{|x-y|^s} \leq \frac{1}{K} \int_{Q_1} \frac{dx}{|x-y|^s} \leq \frac{1}{K} \int_{Q_1} \frac{dx}{|x|^s} = 1. \quad (5)$$

Construct the auxiliary function

$$V(x) = m \left(1 - U(x) + \frac{\operatorname{mes} H}{K \cdot 3^s} \right), \quad \text{where } m = \max_{\bar{D}} u.$$

We have $LV \geq 0$ outside H . Further, on the boundary of the domain D the function V is not less than u . Indeed, the boundary of D consists of Γ (where $u = 0$, while V is positive) and of points belonging to S_4 . But $u|_{S_4} \leq m$, and

$$U|_{S_4} = \frac{1}{K} \int_H \frac{dy}{|x-y|^s} \Big|_{S_4} \leq \frac{\operatorname{mes} H}{K \cdot 3^s},$$

and therefore $V|_{S_4} \geq m$. Hence, by the maximum principle, $V \geq u$ in D , and

$$\max_{\bar{D} \cap Q_1} u \leq \max_{\bar{D} \cap Q_1} V \leq m \left(1 + \frac{\operatorname{mes} H}{K \cdot 3^s} - \min_{Q_1} U \right) \leq m \left[1 + \frac{1}{K} \left(\frac{1}{3^s} - \frac{1}{2^s} \right) \operatorname{mes} H \right],$$

which proves the lemma, since from (4) it follows that K depends on s and n .

With the aid of Lemma 1 we prove

Lemma 2. Let $R > 0$ be arbitrary. Let D be a domain lying inside Q_{2R} , intersecting Q_R , and having boundary points on the boundary of the ball Q_{2R} . Denote by Γ that part of the boundary of the domain D which is situated strictly inside Q_{2R} . Let $u(x)$ be a solution of equation (1) of Cordes type, positive in D , continuous in $\bar{D}$, and vanishing on Γ .

There exists a constant $\delta > 0$, depending on s and n , such that from the inequality $\text{mes } D / \text{mes } Q_R < \delta$ it follows that

$$\max_{\bar{D}} u \geq 2^{n+1} \max_{\bar{D} \cap Q_R} u.$$

Theorem 1 (Harnack inequality). Let equation (1) of Cordes type be defined in the ball Q_{16} . Let $u(x)$ be a solution of this equation, positive in the ball Q_{16} . Then

$$\max_{|x| \leq 1} u(x) / \min_{|x| \leq 1} u(x) < C,$$

where C is a constant depending on s and n .

We give here the main points of the proof. Put $\max_{|x| \leq 1} u(x) = m$ and $v(x) = \frac{2}{m}u(x)$. We have to prove that there exists a constant $\gamma > 0$, depending on s and n , such that

$$\min_{|x| \leq 1} v(x) > \gamma. \quad (6)$$

Denote by D_1 the set of points $x \in Q_4$ where $v(x) > 1$. We consider separately two cases: 1) $\text{mes } D_1 > \delta \omega_n / 4^n$, where δ is the constant of Lemma 2 and ω_n is the volume of the unit n -dimensional sphere, and 2) $\text{mes } D_1 \leq \delta \omega_n / 4^n$.

In the first case the desired inequality (6) is easily obtained by applying Lemma 1 to the function $w(x) = 1 - v(x)/4$. We shall consider the second case.

Denote by S_{r_1, r_2} the spherical layer $r_1 \leq |x| \leq r_2$. Let ρ_1 be the largest of the numbers ρ , $0 < \rho < 1$, such that

$$\text{mes}(D_1 \cap S_{1, 1+\rho}) \geq \delta \omega_n (\rho/4)^n. \quad (7)$$

Since for ρ sufficiently close to zero this inequality is satisfied, while for $\rho = 1$ the reverse inequality holds, such a ρ_1 exists. Put $G_1 = D_1 \cap S_{1, 1+\rho_1}$. Then $\text{mes } G_1 = \delta \omega_n (\rho/4)^n$.

Consider the function $w_1(x) = v(x) - 1$. It is positive on G_1 , nonpositive on $S_{1, 1+\rho_1} \setminus G_1$, and the intersection of G_1 with the sphere $|x| = 1 + \rho_1/2$ contains a point x^1 where $w_1(x^1) > 1$. Denote by Q^1 the ball of radius $\rho_1/2$ with center

at the point x^1 , and by $Q_{1/2}^1$ the concentric ball of half the radius. Denote by D^1 that component of the intersection $G_1 \cap Q^1$ which contains the point x^1 . Applying Lemma 2 to Q^1 , $Q_{1/2}^1$, and D^1 , and then the maximum principle, we find

$$\max_{|x|=1+\rho_1} v(x) \geq \max_{\overline{Q}^1} v > \max_{\overline{Q}^1} w \geq 2 \cdot 2^n.$$

Now denote by D_2 the set of those $x \in Q_4$ where $v > 2^n$. Let ρ_2 be the largest of the numbers ρ , $0 < \rho < 1$, such that

$$\text{mes}(D_2 \cap S_{1+\rho_1, 1+\rho_1+\rho}) \geq \delta \omega_n (\rho/4)^n.$$

Put $G_2 = D_2 \cap S_{1+\rho_1, 1+\rho_1+\rho_2}$, $w_2(x) = v(x) - 2^n$. Then $\text{mes } G_2 = \delta \omega_n (\rho/4)^n$. We have $w_2(x) > 0$ on G_2 , $w_2(x) \leq 0$ on $S_{1+\rho_1, 1+\rho_1+\rho_2}$. At the intersection of G_2 with the sphere $|x| = 1 + \rho_1 + \rho_2$ there is a point x^2 where $w_2(x^2) > 2^n$. Denoting by Q^2 and $Q_{1/2}^2$ the balls with center at x^2 of radii $\rho_2/2$ and $\rho_2/4$, respectively, and by D^2 that component of the intersection $G_2 \cap Q^2$ which contains x^2 , and applying Lemma 2 and the maximum principle, we find

$$\max_{|x|=1+\rho_1+\rho_2} v(x) \geq \max_{\overline{Q}^2} v > \max_{\overline{Q}^2} w_2 \geq 2 \cdot 2^{2n}.$$

We shall continue this process until, for the first time, it becomes true that $\rho_1 + \rho_2 + \dots + \rho_k \geq 1$. This must necessarily occur at a finite step, since otherwise the function $v(x)$ would grow to infinity.

Thus, suppose k steps have been carried out, and $\rho_1 + \dots + \rho_k \geq 1$. For each i , $i = 1, \dots, k$, there is a set G_i such that

$$\text{mes } G_i = \delta \omega_n (\rho_i/4)^n, \quad v(x) > 2^{n(i-1)} \quad \text{on } G_i.$$

Among the numbers $\rho_1, \dots, \rho_k$ there is one ρ_{i_0} such that $\rho_{i_0} > 1/2^{i_0}$. Therefore, in the spherical layer $Q_3 \setminus Q_1$ there is a set G_{i_0} such that

$$\text{mes } G_{i_0} > \frac{\delta \omega_n}{4^n} \frac{1}{2^{ni_0}}, \quad v > 2^{n(i-1)} \quad \text{on } G_{i_0}.$$

Put

$$W(x) = \frac{2^{ni_0}}{2^{nK}} \int_{G_{i_0}} \frac{dy}{|x-y|^s} - \frac{2^{ni_0}}{K \cdot 12^s} \text{mes } G_{i_0}, \quad K = \int_{Q_3} \frac{dx}{|x|^s}.$$

Consider on the set $\Omega = Q_{16} \setminus G_{i_0}$ the functions $W(x)$ and $v(x)$. The boundary Ω consists of the surface S_{16} of the ball Q_{16} and of the boundary Γ_{i_0} of the set G_{i_0} . We have

$$v(x)|_{\Gamma_{i_0}} \geq 2^{n(i-1)}; \quad W(x)|_{\Gamma_{i_0}} \leq 2^{n(i_0-1)};$$

$$v(x)|_{S_{16}} \geq 0; \quad W(x)|_{S_{16}} < \frac{2^{ni_0}}{2^{nK}} \left(\max_{\substack{|x|=16 \\ y \in Q_3}} \frac{1}{|x-y|^s} - \frac{1}{12^s} \right) \text{mes } G_{i_0} < 0.$$

Consequently, by the maximum principle, $v \geq W$ in Ω . Therefore

$$v(x)|_{Q_1} \geq W(x)|_{Q_1} > \frac{2^{ni_0}}{2^{nK}} \left(\min_{\substack{x \in Q_1 \\ y \in Q_3}} \frac{1}{|x-y|^s} - \frac{1}{12^s} \right) \text{mes } G_{i_0} > \frac{\delta \omega_n}{8^{nK}} \left(\frac{1}{4^s} - \frac{1}{12^s} \right) = \gamma,$$

where γ , thus, depends on s and n .

Remark. Since equation (1) is unchanged under a similarity transformation, in the statement of the theorem the absolute values of the radii of the outer and inner balls (16 and 1) are not essential; what matters only is

to their ratio; of course, this ratio can be made arbitrary by changing the constant C in the corresponding way. The same remark also applies to Theorem 2, proved below.

For the validity of Harnack's inequality it is not necessary that equation (1) be of Cordes type in the whole ball Q_{16} . It is sufficient that it be of Cordes type in a layer adjacent to its boundary.

Theorem 2 (Harnack's inequality). *Let an elliptic equation (1) be defined in the ball Q_{16} . Suppose that outside the ball Q_1 it is of Cordes type. Let $u(x)$ be a positive solution of this equation in Q_{16} . Then*

$$\max_{\overline{Q_1}} u(x) / \min_{\overline{Q_1}} u(x) < C,$$

where the constant C depends on s and n .

Proof. Denote by S_8 the sphere $|x| = 8$. Denote by k the least natural number such that for any two points a and b lying on S_8 , one can find a chain of k points $x^1, \dots, x^k$, also lying on S_8 , such that $x^1 = a$, $x^k = b$ and $|x^{i+1} - x^i| < 7/16$, $i = 1, \dots, k-1$.

The number k , obviously, depends on the dimension n of the space. We then have

$$\max_{x \in S_8} u(x) / \min_{x \in S_8} u(x) < C^{k-1},$$

where C is the constant in the Harnack inequality proved in the preceding theorem. Indeed, let the minimum on the sphere S_8 be attained at the point a , and the maximum at the point b . Construct the chain $a = x^1, x^2, \dots, x^k = b$ mentioned above. Applying the proved inequality successively to balls of radius 7 with centers at the points $x^i, i = 1, \dots, k-1$, we obtain $u(b) < C^{k-1}u(a)$.

It remains now for us to note that, by the maximum principle,

$$\max_{\bar{Q}_1} u / \min_{\bar{Q}_1} u \leq \max_{S_8} u / \min_{S_8} u.$$

Theorem 2 gives us the possibility of obtaining Liouville's theorem for elliptic equation (1) under the assumption that it is of Cordes type in a neighborhood of infinity.

Theorem 3 (Liouville). *Let an elliptic equation (1) be defined in the entire space R_n and, outside a ball of some radius R , be of Cordes type. Let $u(x)$ be a solution of this equation, defined in the entire space and nonnegative. Then $u(x) \equiv \text{const}$.*

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Note: Figure translations are in progress. See original paper for figures.

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