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Abstract

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MATHEMATICS

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ON SOME SYSTEMS OF SYMMETRIC FUNCTIONS

We retain the notation of (1): x_i are Cartesian coordinates (c.c.) in E_n ; e_j , $j = 0, 1, \dots, n$, are unit vectors in E_n , the vertices of the regular n -dimensional simplex D_n ; y_j , $j = 0, 1, \dots, n$, are the "barycentric" coordinates (b.c.) of the vector $Y = [y_j] = \sum_j y_j e_j$ in E_n ; $[y_j] = [y_j + y]$, y arbitrary;

$$y_{cp} = \frac{1}{n+1} \sum y_j, \quad y'_j = y_j - y_{cp}$$

are the canonical b.c. of the vector $[y_j] = [y'_j]$; Ω_n is the lattice in E_n , the set of vectors $K = [k_j]$ with integral b.c. k_j ; W_n is the Dirichlet domain of Ω_n , its volume $|W_n| = (n+1)^{(n-1)/2} n^{-n/2}$;

$$\sum_j = \sum_{j=0}^n; \quad \prod_j = \prod_{j=0}^n; \quad \sum_m = \sum_{m=-\infty}^{\infty}; \quad \frac{1}{m!} = 0 \quad \text{for } m < 0.$$

Let $K = [k_j] \subset \Omega_n$

$$U_K^{(n)}(x) = U_{(k_j)}^{(n)}(x) = U_{(k_0, k_1, \dots, k_n)}^{(n)}(x) = \sum_m \prod_j \frac{x^{m+k_j}}{(m+k_j)!}. \quad (1)$$

For $Z = [z_j] = [z'_j]$, $Y = [y_j] = [y'_j]$,

$$\langle Z, Y \rangle = \sum_j z_j y'_j = \sum_j y_j z'_j.$$

1. Let there be given m indices i (for example, $i = 0, 1, \dots, m-1$ or $i = 1, 2, \dots, m$) and m numbers a_i with indices i ; let s be a permutation of these indices, $i \rightarrow si$. Then $sa_i = a_{si}$; S_m is the group of the $m!$ permutations s . If, among the numbers a_i , l_p are equal to p , $\sum_p l_p = m$, then $a(a_i) = \prod_p l_p!$ is the number of permutations $s \in S_m$ for which all $sa_i = a_i$. We shall, in particular, consider the group S_{n+1} of permutations s of the $(n+1)$

indices $j = 0, 1, \dots, n$ of the b.c. of the vector $Y = [y_j]$; we denote $sY = [sy_j] = [y_{sj}]$; thus s may be regarded as a transformation of E_n : $Y \rightarrow sY$, and the group S_{n+1} as the group of transformations of the “simplex D_n ” of the space E_n . Each vector $K = [k_j] \in \Omega_n$ passes under transformations $s \in S_{n+1}$ into vectors sK , differing only in the order of the b.c.; the number of distinct ones among them is $(n+1)!/a(k_j)$; we denote the set of these vectors by $\mathcal{K} = \{k_j\}$; $a(\mathcal{K}) = a(k_j)$, and the set of all $\mathcal{K}$ ’s by $\tilde{\Omega}_n$.

2. The functions $U_K^{(n)}(x) = U_{(k_1 \dots k_n)}(x)$ are symmetric in the k_j . We shall denote by $U_K^{(n)}(x)$ all functions $U_K^{(n)}(x)$ that are equal to one another, $K \in \mathcal{K}$.

Let $\prod_j t_j = 1$, $K = [k_j]$, $t^K = \prod_j t_j^{k_j}$; for $K \in \mathcal{K} = \{k_j\}$,

$$A_n(\mathcal{K}, t) = A_n(t_j, k_j) = \sum_{s \in S_{n+1}} t^{sK}. \quad (2)$$

The generating formula for $U_K^{(n)}(x)$ from (1) takes the form

$$\exp \left(x \sum_j t_j \right) = \sum_{K \in \tilde{\Omega}_n} \frac{1}{a(\mathcal{K})} A_n(\mathcal{K}, t) U_K^{(n)}(x). \quad (3)$$

For $t_j = e^{y'_j}$ ($\sum_j y'_j = 0$, $[y'_j] = Y'$)

$$B_n(\mathcal{K}, Y') = B_n(k_j, y'_j) = A_n(k_j, t_j) = \sum_{s \in S_{n+1}} e^{\langle sK, Y' \rangle}, \quad (4)$$

$$C_n(\mathcal{K}, Y) = B_n(\mathcal{K}, 2\pi i Y). \quad (5)$$

Replacing t_j in (3) by $e^{y'_j}$ ($e^{2\pi i y_j}$) leads to replacing $A_n(\mathcal{K}, t)$ by $B_n(\mathcal{K}, Y)$ ($C_n(\mathcal{K}, Y)$). The integral representation from (1) for $U_K^{(n)}(x)$ takes the form

$$U_K^{(n)}(x) = \frac{1}{|d_n|} \int_{d_n} \dots \int \exp \left(\sum_j x e^{2\pi i y_j} \right) \overline{C_n(\mathcal{K}, Y)} d\omega_n; \quad (6)$$

d_n is the set of points W_n for which $y_0 \leq y_1 \leq \dots \leq y_n$; $|d_n| = \frac{1}{(n+1)!} |W_n|$.

Note that the system $C_n(\mathcal{K}, Y)$, $\mathcal{K} \in \Omega_n$, as functions of Y , forms a complete orthogonal system on d_n , and every function $f(Y)$ with integrable square on d_n is expanded there in the Fourier series

$$f(Y) = \sum_{K \subset \Omega_n} \frac{C_K}{a(K)} C_n(K, Y), \quad C_K = \frac{1}{|d_n|} \int_{d_n} \dots \int f(Y) \overline{C_n(K, Y)} d\omega_n. \quad (7)$$

Remark. Every function $f(Y)$ defined on d_n is extended by the equalities $f(sY) = f(Y)$, $s \in S_{n+1}$, $f(Y + e_j) = f(Y)$ to a function symmetric with respect to the b.c. in E_n with vector periods e_j (such are $C_n(K, Y)$).

3. We consider o.f.–homogeneous forms and s.f.–symmetric o.f. in E_n ; when passing from d.c. to b.c. and conversely, the homogeneity and degree of an o.f. do not change. We distinguish s.f.d. and s.f.b.–s.f. with respect to d.c. and b.c. in E_n . Sph.f.–spherical functions—are o.f. satisfying Laplace's equation (simultaneously with respect to d.c. and b.c.); s.sph.d. (and s.sph.b.) are simultaneously sph.f. and s.f.d. (s.f.b.). $[o.f._{p,n}]$, $[s.f.b._{p,n}]$, etc. are linear systems of all o.f. (s.f.b., etc.) in E_n of degree p ; their dimensions are $\dim[o.f._{p,n}]$, etc. In what follows p, n, m, l, p_i are natural numbers.

We consider representations of p by sums

$$p = \sum_{i=1}^l p_i$$

under restrictions on p_i and l : a) $p_i \geq m$; b) $p_i \leq n$; c) $p_i \neq k$; d) $l \leq n$.

We denote the number of such representations of p under the conditions: a) by $\alpha(p, m)$; a) and b) by $\alpha(p, m, n)$; a), b), and c) by $\alpha(p, m, n, k)$; a) and d) by $\beta(p, m, n)$.

Obviously, for $n \geq p$, $\alpha(p, m) = \alpha(p, m, n) = \beta(p, m, n)$. Further, $\beta(p, 1, n) = \alpha(p, 1, n)$, $\beta(p, m, n) > \alpha(p, m, n)$ for $m > 1$, $n < p$; $\alpha(p, m, n, k) = \alpha(p, m, n) - \alpha(p - k, m, n)$.

As $p \rightarrow \infty$, $m < n$,

$$\alpha(p, m, n) = \frac{(m-1)!}{n!(n-m)!} p^{n-m} + O(p^{n-m-1}).$$

$$\dim[s.f.d._{p,n}] = \alpha(p, 1, n); \quad \dim[s.f.b._{p,n}] = \alpha(p, 2, n+1);$$

$$\dim[s.sph.d._{p,n}] = \alpha(p, 1, n, 2); \quad \dim[s.sph.b._{p,n}] = \alpha(p, 3, n+1),$$

$$\lim_{p \rightarrow \infty} \frac{\dim[s.f.d._{p,n}]}{\dim[o.f._{p,n}]} = \lim_{p \rightarrow \infty} \frac{\dim[s.sph.d._{p,n}]}{\dim[sph.f._{p,n}]} = \frac{1}{n!},$$

$$\lim_{p \rightarrow \infty} \frac{\dim[\text{s.f.b.}_{p,n}]}{\dim[\text{o.f.}_{p,n}]} = \lim_{p \rightarrow \infty} \frac{\dim[\text{s.sph.b.}_{p,n}]}{\dim[\text{sph.f.}_{p,n}]} = \frac{1}{(n+1)!}.$$

All symmetric functions of m variables $z_1, z_2, \dots, z_m$ of degree p are linear combinations of the simplest such symmetric functions $T_{(r_i)}^{(m)} = T_{r_1, r_2, \dots, r_m}^{(m)}$, “generated” by the products $\prod_i z_i^{r_i}$:

$$T_{(r_i)}^{(m)}(z_i) = \frac{1}{a(r_i)} \sum_{s \in S_m} \prod_i (sz_i)^{r_i}, \quad \sum_i r_i = p, \quad r_i \geq 0.$$

In [s.f. $d_{n,p}$] the forms $T_{(r_i)}^{(n)}$ in the variables x_i , $\sum_i r_i = p$, form a basis.

In [s.f. $b_{n,p}$] all forms are expressed linearly in terms of $T_{(r_j)}^{(n+1)}(y'_j)$, $\sum_j r_j = p$, where $r_j \neq 1$; for $n+1 \geq p$ they form a basis, while for $n+1 < p$ there exist $\beta(p, 2, n+1) - \alpha(p, 2, n+1) > 0$ linear relations among them. For example, in [s.f. $b_{n,5}$] for $n \geq 4$ the basis is formed by $T_5^{(n+1)}(y'_j) = \sum_j y_{j'}^5$ and $T_{3,2}^{(n+1)} = \sum_{i \neq j} y_i^3 y_{j'}^2$ (we omit the indices $r_j = 0$ in the notation $T_{(r_j)}^{(n+1)}$). If $n < 4$, $(T_5^{(n+1)} - 5T_{3,2}^{(n+1)})(y'_j) = 0$; for $n = 1$, $T_5^{(n+1)} = T_3^{(n+1)} = 0$.

The symmetrized exponential functions $B_n(\mathcal{K}, Y) = B_n(k_j, y'_j)$ are expanded in a series in the s.f.b. ($[T]^2 = T(k'_j)T(y'_j)$)

$$B^{(n)}(\mathcal{K}, Y) = (n+1)! + \sum_{p=2}^{\infty} a_{p,n}(k_j, y'_j),$$

$$a_{p,n}(k_j, y'_j) = \sum_{r_j \geq 0} \dots \sum_{\sum_j r_j = p} \frac{a(r_j) p!}{\prod_j r_j!} [T_{(r_j)}^{(n+1)}]^2. \quad (8)$$

Expressing all $T_{(k_j)}^{(n+1)}(y'_j)$ in terms of the basis functions, we obtain for $p = 2, 3, 4, 5$ (when $n+1 < p$ the terms containing the factor $(n-k)!$, where $k > n$, vanish):

$$a_{2,n}(k_j, y'_j) = (n+1)(n-1)! [T_2^{(n+1)}]^2,$$

$$a_{3,n}(k_j, y'_j) = (n+1)^2 (n-2)! [T_3^{(n+1)}]^2,$$

$$a_{4,n}(k_j, y'_j) = (n+4)(n-1)! [T_4^{(n+1)}]^2 + 12(n-1)! [T_{2,2}^{(n+1)}]^2 + (6n-3)(n-3)! [T_4^{(n+1)} - 2T_{2,2}^{(n+1)}]^2,$$

$$a_{5,n}(k_j, y'_j) = (n-2)! \{ (n+5)(n-1)[T_5^{(n+1)}]^2 + 10(n+5)[T_{3,2}^{(n+1)}]^2 \\ + 10[T_5^{(n+1)} - T_{3,2}^{(n+1)}]^2 \} + (10n-14)(n-4)! [T_5^{(n+1)} - 5T_{3,2}^{(n+1)}]^2.$$

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1. L. A. Lyusternik, *DAN*, **177**, No. 5 (1967).

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