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Abstract

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MATHEMATICS

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ON A CERTAIN FAMILY OF CONDITIONALLY PERIODIC SOLUTIONS OF A HAMILTONIAN SYSTEM

(Presented by Academician A. N. Kolmogorov on 22 IV 1968)

Consider the Hamiltonian system

$$\dot{x} = -\partial H / \partial y, \quad \dot{y} = \partial H / \partial x \quad (x = (x_1, \dots, x_m), \quad y = (y_1, \dots, y_m)),$$

$$\dot{p} = -\partial H / \partial q, \quad \dot{q} = \partial H / \partial p \quad (p = (p_1, \dots, p_n), \quad q = (q_1, \dots, q_n)) \quad (1)$$

with a Hamiltonian function 2π -periodic in q , and suppose that system (1) has a conditionally periodic solution

$$x = f(\varphi), \quad y = g(\varphi), \quad p = r(\varphi), \quad q = \varphi + s(\varphi), \quad \dot{\varphi} = \omega, \quad (2)$$

where f, g, r, s are 2π -periodic functions of φ , and $\omega = (\omega_1, \dots, \omega_n)$ are constants. Suppose further that f, g, r, s are analytic functions of φ in the strip $|\operatorname{Im} \varphi| < \rho$ ($\rho > 0$), the determinant $\det |E_n + \partial s / \partial \varphi|$ is different from zero in this strip, and the function H is analytic in x, y, p, q in some neighborhood of the solution (2). Under these conditions the change of variables

$$x = f(\varphi) + \xi, \quad y = g(\varphi) + \eta,$$

$$p = r(\varphi) + \left(E_n + \frac{\partial s}{\partial \varphi}\right)^{-1} \left(I - \frac{\partial g}{\partial \varphi} \xi + \frac{\partial f}{\partial \varphi} \eta\right), \quad q = \varphi + s(\varphi)$$

brings system (1) to the form

$$\dot{\xi} = -\partial \mathcal{H} / \partial \eta, \quad \dot{\eta} = \partial \mathcal{H} / \partial \xi, \quad \dot{I} = -\partial \mathcal{H} / \partial \varphi, \quad \dot{\varphi} = \partial \mathcal{H} / \partial I,$$

where

$$\mathcal{H} = H \left(f(\varphi) + \xi, g(\varphi) + \eta, r(\varphi) + \left(E_n + \frac{\partial s}{\partial \varphi} \right)^{-1} \left(I - \frac{\partial g}{\partial \varphi} \xi + \frac{\partial f}{\partial \varphi} \eta \right) \varphi + s(\varphi) \right),$$

and, according to the equalities (1), (2), for $\xi = \eta = I = 0$ we have $\partial \mathcal{H} / \partial \xi = \partial \mathcal{H} / \partial \eta = 0$, $\partial \mathcal{H} / \partial I = \omega$.

Let us now take, as the starting point of the considerations given above, any one of the conditionally periodic solutions obtained in work ⁽¹⁾. In this case, on the basis of the results of work ⁽¹⁾, there exists a canonical change of variables

$$\xi = \partial V / \partial \eta, \quad v = \partial V / \partial u, \quad I = \partial V / \partial \varphi, \quad \psi = \partial V / \partial z \quad (3)$$

with a generating function V of the form

$$V = u\eta + z\varphi + \frac{1}{2}\alpha(\varphi)u^2 + \beta(\varphi)u\eta + \frac{1}{2}\gamma(\varphi)\eta^2 + \chi(\varphi)uz + \tau(\varphi)\eta z + \frac{1}{2}\chi(\varphi)z^2 \quad (4)$$

such that the matrices $\alpha, \beta, \gamma, \kappa, \tau, \chi$ are 2π -periodic functions of φ , analytic in φ in the strip $|\operatorname{Im} \varphi| < \rho'$ ($\rho' > 0$), the determinant $\det |E_m + \beta|$ is different from zero in this strip, and the expansion of $\mathcal{H}$ in a series in powers of u, v, z has the form

$$\mathcal{H} = \mathcal{H}_0 + \omega z + \frac{1}{2}au^2 + buv + \frac{1}{2}cv^2 + \frac{1}{2}dz^2 + \dots, \quad (5)$$

where the dots denote terms of higher order, and $\mathcal{H}_0$ and the matrices a, b, c, d do not depend on ψ .

Suppose now that all eigenvalues of the matrix

$$\begin{vmatrix} -b' & -c \\ a & b \end{vmatrix} \quad (6)$$

are simple, and among them there are $2l$ ($1 \leq l \leq m$) purely imaginary eigenvalues $\lambda_1 = i\nu_1, \dots, \lambda_l = i\nu_l, \lambda_{m+1} = -i\nu_1, \dots, \lambda_{m+l} = -i\nu_l$. In this case the canonical change of variables (3) with generating function of the form (4) can be chosen so that the expansion (5) takes the form

$$\mathcal{H} = \mathcal{H}_0 + \omega z + \frac{1}{2}\nu\tilde{u}^2 + \frac{1}{2}A_0\tilde{u}^2 + B_0\tilde{u}\tilde{v} + \frac{1}{2}C_0\tilde{v}^2 + \frac{1}{2}\nu\tilde{v}^2 + dz^2 + \dots, \quad (7)$$

where $\hat{u} = (u_1, \dots, u_l)$, $\hat{v} = (v_1, \dots, v_l)$, $\tilde{u} = (u_{l+1}, \dots, u_m)$, $\tilde{v} = (v_{l+1}, \dots, v_m)$, $\nu = (\nu_1, \dots, \nu_l)$, and the matrices A_0, B_0, C_0 do not depend on ψ .

For most of the conditionally periodic solutions obtained in [1] there exists a constant $K > 0$ such that, for any integer $k = (k_1, \dots, k_n) \neq 0$, the inequalities

$$\left| \sum_{j=1}^5 \lambda_{\alpha_j} + i(\omega, k) \right| \geq \frac{K}{|k|^{n+1}}, \quad \left| \sum_{j=1}^6 \lambda_{\alpha_j} + i(\omega, k) \right| \geq \frac{K}{|k|^{n+1}}, \quad (8)$$

hold, where $|k| = |k_1| + \dots + |k_n|$, $(\omega, k) = \omega_1 k_1 + \dots + \omega_n k_n$, and the λ_{α_j} are arbitrary eigenvalues of the matrix (6). Proceeding from this, assume that, for the solution (2), the inequalities (8) are satisfied. Suppose, further, that the sum $\sum_{j=1}^5 \lambda_{\alpha_j}$ is different from zero for all $\alpha_j = 1, \dots, 2m$, while the sum $\sum_{j=1}^6 \lambda_{\alpha_j}$ vanishes only in the case when the eigenvalues λ_{α_j} are combined pairwise so that the sum of each pair is zero. Hence, taking account of equality (7), it follows that there exists a canonical change of variables $u = \partial V / \partial v$, $y = \partial V / \partial x$, $z = \partial V / \partial \psi$, $q = \partial V / \partial p$ with generating function V of the form

$$V = xv + p\psi + \sum_{s=3}^6 V_s(x, v, p, \psi),$$

where V_s is a homogeneous polynomial of degree s in the variables x, v, p , whose coefficients are 2π -periodic functions of ψ , analytic in ψ in the strip $|\operatorname{Im} \psi| < \rho'$, such that in the new variables the equality

$$\mathcal{H} = \sum_{s=0}^6 \mathcal{H}_s(p, \tau) + \frac{1}{2} \sum_{s=0}^4 \{A_s(p, \tau) \tilde{x}^2 + 2B_s(p, \tau) \tilde{x} \tilde{y} + C_s(p, \tau) \tilde{y}^2\} + P_4(\tilde{x}, \tilde{y}, p, \tau) + \mathcal{H}^*(x, y, p, q), \quad (9)$$

holds, where $2\tau = (x_1^2 + y_1^2, \dots, x_l^2 + y_l^2)$; $\tilde{x} = (x_{l+1}, \dots, x_m)$; $\tilde{y} = (y_{l+1}, \dots, y_m)$; $\mathcal{H}_s, A_s, B_s, C_s$ are homogeneous polynomials of degree s in the variables p, τ ; the polynomial P_4 contains no powers of $\tilde{x}, \tilde{y}$ lower than the fourth, and the expansion of $\mathcal{H}^*$ in a series in powers of x, y, p contains no terms lower than the seventh degree. Proceeding from equality (9), put

$$\mathcal{H} = \bar{H}(J) + \frac{1}{2} A(J) z^2 + B(J) zw + \frac{1}{2} C(J) w^2 + H^*(z, w, J, \theta), \quad (10)$$

where $z = \tilde{x}$, $w = \tilde{y}$, $J = (p, \tau)$, $\theta = (q, \sigma)$,

$$\sigma = \left(\arctg \frac{y_1}{x_1}, \dots, \arctg \frac{y_l}{x_l} \right),$$

$$\bar{H} = \sum_{s=0}^6 \mathcal{H}_s, \quad A = \sum_{s=0}^4 A_s, \quad B = \sum_{s=0}^4 B_s, \quad C = \sum_{s=0}^4 C_s.$$

In this situation the following theorem holds.

Theorem. Suppose a Hamiltonian system is given

$$\dot{z} = -\partial \mathcal{H} / \partial w, \quad \dot{w} = \partial \mathcal{H} / \partial z \quad (z = (z_1, \dots, z_{m'}), \quad w = (w_1, \dots, w_{m'})), \quad (11)$$

$$\dot{J} = -\partial \mathcal{H} / \partial \theta, \quad \dot{\theta} = \partial \mathcal{H} / \partial J \quad (J = (J_1, \dots, J_{n'}), \quad \theta = (\theta_1, \dots, \theta_{n'}))$$

with Hamiltonian function (10), satisfying the following conditions:

1. $\mathcal{H}$ is a 2π -periodic function of θ , analytic in z, w, J, θ in a domain of the form

$$|z, w| \leq R_0, \quad |J_1| \leq \varepsilon, \dots, \quad |J_n| \leq \varepsilon, \quad |J_{n+1} - 2\varepsilon| \leq \varepsilon, \dots, \\ \dots, \quad |J_{n'} - 2\varepsilon| \leq \varepsilon, \quad |\operatorname{Im} \theta| < \rho_0 \quad (R_0 > 0, \quad \rho_0 > 0, \quad 0 < \varepsilon < \varepsilon_0, \quad 0 \leq n \leq n').$$

2. For $0 < \varepsilon < \varepsilon_0$ and $z = w = 0$, in the domain

$$|J_1| \leq \varepsilon, \dots, \quad |J_n| \leq \varepsilon, \quad |J_{n+1} - 2\varepsilon| \leq \varepsilon, \dots, \quad |J_{n'} - 2\varepsilon| \leq \varepsilon, \quad |\operatorname{Im} \theta| \leq \rho_0$$

the inequalities hold

$$|H^*| \leq \varepsilon^3 \mu, \quad |\partial H^* / \partial \chi| \leq \varepsilon^{5/2} \mu, \quad |\partial^2 H^* / \partial \chi^2| \leq \varepsilon^2 \mu, \quad |\partial^3 H^* / \partial \chi^3| \leq \varepsilon^{3/2}, \quad (12)$$

where $\chi = (z, w)$ and $\mu = O(\sqrt{\varepsilon})$.

3. H, A, B, C are analytic functions of J in some neighborhood of $J = 0$, and for $J = 0$

$$\det |\partial^2 H / \partial J^2| \neq 0, \quad (13)$$

and all eigenvalues $\lambda_\alpha = \lambda_\alpha(J)$ of the matrix

$$\Gamma = \begin{vmatrix} -B' & -C' \\ A & B \end{vmatrix}$$

for $J = 0$ are simple and satisfy an inequality of the form

$$\sum_{s=1}^{n'} \left(\frac{\partial \lambda_\alpha}{\partial J_s} + \frac{\partial \lambda_\beta}{\partial J_s} + i \sum_{r=1}^{n'} k_r \frac{\partial^2 H}{\partial J_r \partial J_s} \right)^2 \neq 0, \quad (13')$$

where k_r are arbitrary integers.

Then there exists a decomposition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ of the domain

$$\mathcal{E} = \{-\varepsilon \leq J_1 \leq \varepsilon, \dots, -\varepsilon \leq J_n \leq \varepsilon, \varepsilon \leq J_{n+1} \leq 3\varepsilon, \dots, \varepsilon \leq J_{n'} \leq 3\varepsilon\}$$

such that $\text{mes } \mathcal{E}_2 / \text{mes } \mathcal{E} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and for any $J^* \in \mathcal{E}_1$ there exists a torus T_{J^*} , invariant with respect to the motions of system (11), possessing the following properties:

1. The invariant tori T_{J^*} are given by parametric equations

$$z = Z_{J^*}(Q), \quad w = W_{J^*}(Q), \quad J = F_{J^*}(Q), \quad \theta = Q + G_{J^*}(Q),$$

where $Z_{J^*}, W_{J^*}, F_{J^*}, G_{J^*}$ are 2π -periodic functions of Q , analytic in Q in the strip

$$|\text{Im } Q| \leq \frac{1}{2}\rho_0.$$

2. The motion on the torus T_{J^*} is conditionally periodic with n' frequencies

$$\omega = (\omega_1, \dots, \omega_{n'}),$$

i.e.

$$\dot{Q} = \omega,$$

where $\omega = \partial H / \partial J$ for $J = J^*$.

The proof of this theorem is based on the following. Let $H_0, H_\chi, H_{\chi\chi}$ denote, respectively, the values of the function H^* and of its partial derivatives with respect to $\chi = (z, w)$ for $z = w = 0$. Put

$$\bar{H} = H + \bar{H}_0, \quad \hat{H}_0 = H_0 - \bar{H}_0, \quad \hat{H}_{\chi\chi} = H_{\chi\chi} - \bar{H}_{\chi\chi},$$

where

$$\bar{H}_0 = (2\pi)^{-n'} \oint H_0(J, \theta) d\theta, \quad \bar{H}_{\chi\chi} = (2\pi)^{-n'} \oint H_{\chi\chi}(J, \theta) d\theta$$

and the integral is taken over the surface of the n' -dimensional torus. We now perform the canonical change of variables

$$z = \partial V / \partial w, \quad w' = \partial V / \partial z', \quad J = \partial V / \partial \theta, \quad \theta' = \partial V / \partial J', \quad (14)$$

with generating function V of the form

$$V = z'w + J'\theta + S_0(J', \theta) + S_z(J', \theta)z' + S_w(J', \theta)w + \frac{1}{2}S_{z'z'}(J', \theta)z'^2 + S_{z'w}(J', \theta)z'w + \frac{1}{2}S_{ww}(J', \theta)w^2,$$

satisfying the system of equations

$$\begin{aligned} (\omega, \partial S_0 / \partial \theta) + [\hat{H}_0(J', \theta)]_N &= 0, \\ (\omega, \partial S_\chi / \partial \theta) + \bar{\Gamma}' S_\chi + [H_\chi(J', \theta)]_N &= 0, \end{aligned} \quad (15)$$

$$(\omega, \partial S_{\chi\chi}/\partial\theta) + \bar{\Gamma}' S_{\chi\chi} + S_{\chi\chi} \bar{\Gamma} + [\hat{H}_{\chi\chi}(J', \theta)]_N + \Delta(J') - \left[\sum_{\alpha=1}^{m'} \left(S_{z_\alpha} \frac{\partial H_{\chi\chi}}{\partial w_\alpha} - S_{w_\alpha} \frac{\partial H_{\chi\chi}}{\partial z_\alpha} \right) \right]_N + \sum_{\alpha=1}^{n'} L \frac{\partial \bar{\Gamma}}{\partial J_\alpha} \frac{\partial S_0}{\partial \theta_\alpha} = 0.$$

where $\omega = \partial \bar{H}/\partial J'$, $\bar{\Gamma} = \Gamma - L \bar{H}_{\chi\chi}$, $S_\chi = (S_{z'}, S_w)$, $\partial H_{\chi\chi}/\partial z_\alpha$ and $\partial H_{\chi\chi}/\partial w_\alpha$ are third-order derivatives of H^* at $z = w = 0$,

$$\Delta = (2\pi)^{-n'} \sum_{\alpha=1}^{m'} \oint \left(S'_{z_\alpha} \frac{\partial H_{\chi\chi}}{\partial w_\alpha} - S_{w_\alpha} \frac{\partial H_{\chi\chi}}{\partial z_\alpha} \right) d\theta,$$

$$L = \begin{vmatrix} 0 & E_{m'} \\ -E_{m'} & 0 \end{vmatrix}, \quad S_{\chi\chi} = \begin{vmatrix} S_{z'z'} & S_{z'w} \\ S_{z'w} & S_{ww} \end{vmatrix},$$

and, for an f of the form $f = \sum_{|k|=0}^{\infty} f_k e^{i(k, \theta)}$, the symbol $[f]_N$ denotes $\sum_{|k| < N} f_k e^{i(k, \theta)}$.

Take the domain $\mathcal{E}^* = \{|J_1| \leq \varepsilon, \dots, |J_n| \leq \varepsilon, |J_{n+1} - 2\varepsilon| \leq \varepsilon, \dots, |J_{n'} - 2\varepsilon| \leq \varepsilon\}$, and suppose that in the domain $\mathcal{E}_N^\delta \subset \mathcal{E}^* - \varepsilon\beta^*$ the inequalities $|(\omega, k)| \geq \varepsilon\delta|k|^{-(n'+1)}$, $|\lambda_\alpha + i(\omega, k)| \geq \varepsilon\delta|k|^{-(n'+1)}$, $|\lambda_\alpha + \lambda_\beta + i(\omega, k)| \geq \varepsilon\delta|k|^{-(n'+1)}$ hold for all integer $k = (k_1, \dots, k_{n'})$ such that $0 < |k| < N$, and for arbitrary eigenvalues of the matrix $\bar{\Gamma}$. Then, according to inequalities (12), for sufficiently small $\beta > 0$, $\delta > 0$, the system of equations (15) has, in the domain $J' \in \mathcal{E}_N^\delta$, $|\operatorname{Im} \theta| \leq \rho_0 - 2\delta$, a solution analytic in this domain and satisfying the inequalities

$$|S_0| \leq \varepsilon^2 \mu \delta^{-\chi}, \quad |S_{z'}, S_w| \leq \varepsilon^{3/2} \mu \delta^{-\chi}, \quad |S_{z'z'}, S_{z'w}, S_{ww}| \leq \varepsilon \mu \delta^{-2\chi},$$

where $\chi = 2n' + 5$. It follows that, for sufficiently small $\varepsilon > 0$ and $\mu\beta^{-1}\delta^{-2(\chi+1)}$, the equalities (14) define a one-to-one change of variables in the domain

$$|z', w'| \leq R_0 - \delta, \quad J' \in \mathcal{E}_N^\delta - 2\varepsilon\beta, \quad |\operatorname{Im} \theta'| \leq \rho_0 - 4\delta.$$

Put now $N = -\ln \mu/\gamma$, $\mu = \delta^{8\chi+2}$. Then, in the new variables, the function $\mathcal{H}$ will have the form

$$\mathcal{H} = \bar{H}(J') + \frac{1}{2}(A(J') + a(J'))z'^2 + (B(J') + b(J'))z'w' + \frac{1}{2}(\bar{C}(J') + c(J'))w'^2 + H'(z', w', J', \theta'),$$

where the matrices a, b, c are connected with the matrix Δ by the equality

$$\Delta = - \begin{vmatrix} a & b \\ b' & c \end{vmatrix}, \quad \bar{\Gamma} = \begin{vmatrix} -\bar{B}' & -\bar{C}' \\ \bar{A} & \bar{B} \end{vmatrix},$$

and the function H' and its derivatives with respect to $\chi' = (z', w')$ at $z' = w' = 0$ in the domain

$$J' \in \mathcal{E}_N^\delta - 3\varepsilon\beta, \quad |\operatorname{Im} \theta'| \leq \rho_0 - 3\gamma, \quad (\gamma > 2\delta > 0)$$

satisfy the inequalities

$$|H'| \leq \varepsilon^3 \mu', \quad |\partial H' / \partial \chi'| \leq \varepsilon^{5/2} \mu', \quad |\partial^2 H' / \partial \chi'^2| \leq \varepsilon^2 \mu',$$

$$|\partial^3 H' / \partial \chi'^3| \leq (1 + \delta) \varepsilon^{1/2} c \mu' = \mu'^{3/2}.$$

On the other hand, from inequalities (13) and (13') it follows that, for $\beta = \delta^3$, $\gamma = \delta^{1/4n'}$, and any fixed $\varepsilon \in (0, \varepsilon_0)$, we have

$$\text{mes}(\mathcal{E} \setminus (\mathcal{E}_N^\delta - 3\varepsilon\beta)) \rightarrow 0$$

as $\delta \rightarrow 0$. Hence it follows that the construction described above can be applied an unlimited number of times. The convergence of this process is proved analogously to how this is done in paper (2).

Inequality (13') in the present paper replaces the requirement used in paper (1) that there be no identical relations of the form

$$\lambda_\alpha + \lambda_\beta + i(\omega, k) \equiv 0$$

with integer $k = (k_1, \dots, k_n)$. In this connection I note that the objections contained in paper (3) against this requirement are apparently based on a misunderstanding. It is essential to observe that, for fixed $\bar{H}$ and $\bar{\Gamma}$, inequality (13') is nontrivial only for a finite number of values of k , since for sufficiently large $|k|$ the validity of inequality (13') follows from inequality (13).

The situation considered in the present paper is rather general. It includes many of the cases previously considered (1-6).

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* The domain $\mathcal{E}^* - \varepsilon\beta$ contains those points of the domain $\mathcal{E}^*$ whose distance to the boundary of the domain $\mathcal{E}^*$ is greater than $\varepsilon\beta$ ($\varepsilon\beta > 0$).

Note: Figure translations are in progress. See original paper for figures.

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