

EMBEDDING THEOREMS FOR WEIGHTED FUNCTION SPACES

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Abstract

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MATHEMATICS

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EMBEDDING THEOREMS FOR WEIGHTED FUNCTION SPACES

(Presented by Academician S. L. Sobolev, 13 III 1968)

In the present paper we consider embedding theorems for the weighted function spaces $W_{p,\alpha}^r(E^n)$, $L_{p,\alpha}^r(E^n)$ into a space with mixed norm. E^n is the set of points $\{x_1, x_2, \dots, x_n\}$ of the n -dimensional Euclidean space E^n for which $x_n > 0$.

Let $f(\bar{x})$ be a smooth function defined in the space E^n . The numbers α , r_i ($i = 1, \dots, n$), and p are real and satisfy the conditions $\alpha > -1$, $0 < r_i < 1$, $1 \leq p \leq \infty$. We shall say that $f \in L_{p,\alpha}^{r_i}(E^n)$ if

$$\|f\|_{L_{p,\alpha}^{r_i}(E^n)} = \left(\int_0^\infty \frac{dt}{t^{1+pr_i}} \int_{E^n} x_n^\alpha |\Delta_i(t)f(\mathbf{x})| d\mathbf{x} \right)^{1/p} < \infty,$$

$$\|f\|_{L_{p,\alpha}^r(E^n)} = \sum_{i=1}^n \|f\|_{L_{p,\alpha}^{r_i}(E^n)},$$

$$\|f\|_{W_{p,\alpha}^r(E^n)} = \sum_{i=1}^n \|f\|_{L_{p,\alpha}^{r_i}(E^n)} + \|f\|_{L_p(E^n)}.$$

The closures of smooth finite functions in the corresponding norms will be denoted by $W_{p,\alpha}^r(E^n)$ and $L_{p,\alpha}^r(E^n)$. We shall say that $f(\mathbf{x}) \in L^{\rho_s}(p, p_1, p_2, \dots, p_m)(E^m)$ if

$$\|f\|_{L^{\rho_s}(p, p_1, p_2, \dots, p_m)(E^m)} = \left(\int_0^\infty \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f(\mathbf{x})\|_{L(p_1, p_2, \dots, p_m)(E^m)}^p \right)^{1/p} < \infty,$$

$$\|f\|_{\tilde{L}^\rho(p, p_1, p_2, \dots, p_m)(E^m)} = \sum_{s=1}^m \|f\|_{L^{\rho_s}(p, p_1, p_2, \dots, p_m)(E^m)},$$

$$\|f\|_{\tilde{W}_{(p,p_1,p_2,\dots,p_m)}^\rho(E^m)} = \|f\|_{L_{(p_1,\dots,p_m)}(E^m)} + \|f\|_{\tilde{L}_{(p,p_1,\dots,p_m)}^\rho(E^m)},$$

where $0 < \rho_s < 1$ ($s = 1, \dots, m$),

$$\|f\|_{L_{(p_1,p_2,\dots,p_m)}(E^m)} = \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \dots \left(\int_{-\infty}^{\infty} |f|^{p_1} dx_1 \right)^{p_2/p_1} \dots \right)^{p_m/p_{m-1}} dx_m \right)^{1/p_m} < \infty.$$

We formulate the results obtained in the form of embedding theorems.

Theorem 1. If $f(\bar{x}) \in W_{p,\alpha}^r(E^n_+)$ and the conditions

$$1 < p \leq p_1 \leq \dots \leq p_m < \infty, \quad 0 < m \leq n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

are satisfied,

$$\varepsilon = 1 + \sum_{i=1}^m \frac{1}{p_i r_i} - \frac{1}{p} \sum_{i=1}^n \frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

then, for $x_n = 0$, $f \in L_{(p_1,p_2,\dots,p_m)}(E^m)$, and the inequality

$$\|f\|_{L_{(p_1,p_2,\dots,p_m)}(E^m)} \leq c \|f\|_{W_{p,\alpha}^r(E^n_+)}$$

holds.

Theorem 2. If $f(x) \in L_{p,\alpha}^r(E^n_+)$ and the conditions

$$1 < p \leq p_1 \leq \dots \leq p_m < \infty, \quad 0 < m \leq n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

$$\varepsilon = 1 + \sum_{i=1}^m \frac{1}{p_i r_i} - \frac{1}{p} \sum_{i=1}^n \frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

are satisfied, then, for $x_n = 0$, $f \in L_{(p_1,p_2,\dots,p_m)}(E^m)$, and the inequality

$$\|f\|_{L_{(p_1, p_2, \dots, p_m)}(E^m)} \leq c \|f\|_{L_{p, \alpha}^r(E^{n+})}$$

holds.

Theorem 3. If $f(x) \in W_{p, \alpha}^r(E^{n+})$ and the conditions

$$1 < p \leq p_1 \leq \dots \leq p_m < \infty, \quad 0 < m \leq n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

$$0 < \rho_s \leq \varepsilon r_s \quad (s = 1, \dots, m), \quad \varepsilon = 1 + \sum_{i=1}^m \frac{1}{p_i r_i} - \frac{1}{p} \sum_{i=1}^n \frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

are satisfied, then, for $x_n = 0$, $f \in L_{(p, p_1, \dots, p_m)}^{\bar{p}}(E^m)$, and the inequality

$$\|f\|_{L_{(p, p_1, \dots, p_m)}^{\bar{p}}(E^m)} \leq c \|f\|_{W_{p, \alpha}^r(E^{n+})}$$

holds.

Theorem 4. If $f(x) \in W_{p, \alpha}^r(E^{n+})$ and the conditions

$$1 < p \leq p_1 \leq \dots \leq p_m < \infty, \quad 0 < m \leq n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

$$0 < \rho_s \leq \varepsilon r_s \quad (s = 1, \dots, m), \quad \varepsilon = 1 + \sum_{i=1}^m \frac{1}{r_i p_i} - \frac{1}{p} \sum_{i=1}^n \frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

are satisfied, then, for $x_n = 0$, $f \in W_{(p, p_1, \dots, p_m)}^{\bar{p}}(E^m)$, and the inequality

$$\|f\|_{W_{(p, p_1, \dots, p_m)}^{\bar{p}}(E^m)} \leq c \|f\|_{W_{p, \alpha}^r(E^{n+})}$$

holds.

In proving these theorems, the integral representation of V. P. Il' in ⁽²⁾ is used.

We shall outline the proof of one of the stated theorems, for example, Theorem 3.

We have

$$\|f\|_{L_{(p_1, p_2, \dots, p_m)}^{\rho_s}(E^m)} = \left(\int_0^\infty \frac{dt}{t^{1-\rho_s p}} \|\Delta_s(t)f(x)\|_{L_{(p_1, p_2, \dots, p_m)}(E^m)}^p \right)^{1/p}; \quad (1)$$

we represent the function $f(x)$ in the form

$$f(x) = \sum_{i=1}^? f_i, \quad (2)$$

where

$$f_1(\bar{x}, h) = \frac{c}{h^\omega} \int_0^{h^{\sum_{j=1}^n \sigma_j}} f(\bar{x} + \bar{y}) \Pi(\bar{y}, h) d\bar{y},$$

$$f_2(\bar{x}, t, h) = -c \sum_{i=1}^n \int_0^{t^{1/\sigma_s}} \frac{dv}{v^{1+\omega}} \int_0^{v^{\sum_{j=1}^n \sigma_j}} d\bar{y} \int_0^{v^{\sigma_i - y_i}} \Delta_i(\tau) f(\bar{x} + \bar{y}) R_i(\bar{y}, \tau, v) d\tau,$$

$$f_3(x, t, h) = -c \sum_{i=1}^n \int_{t^{1/\sigma_s}}^h \frac{dv}{v^{1+\omega}} \int_0^{v^{\sum_{j=1}^n \sigma_j}} d\bar{y} \int_0^{v^{\sigma_i - y_i}} \Delta_i(\tau) f(\bar{x} + \bar{y}) R_i(\bar{y}, \tau, v) d\tau,$$

where $\sigma_j = 1/r_j$.

Taking equality (2) into account, we have

$$\|\Delta_s(t)f(x)\|_{L_{(p_1, p_2, \dots, p_m)}(E^m)} \leq \sum_{i=1}^3 \|\Delta_s(t)f_i\|_{L_{(p_1, p_2, \dots, p_m)}(E^m)}. \quad (3)$$

The following estimates hold:

$$\|\Delta_s(t)f_1(\bar{x}h)\|_{L_{(p)}(E^m)} \leq cth^{-\frac{1}{p} \sum_{j=1}^n \sigma_j + \sum_{j=1}^m \frac{\sigma_j}{p_j} - \sigma_s} \|f\|_{L_p^+(E^n)}; \quad (4)$$

$$\begin{aligned} \|\Delta_s(t)f_2(\bar{x}, t, h)\|_{L_{(p)}(E^m)} &\leq c \sum_{i=1}^n \left\| \int_0^{t^{1/\sigma_s}} \frac{dv}{v^{1+\lambda_i - \varepsilon + \gamma \sigma_i - \alpha/pr_n}} \times \right. \\ &\times \int_0^{v^{\sigma_i}} \tau^\gamma d\tau \int_x^{x+v^{\sum_{j=1}^n \sigma_j}} \frac{|\Delta_i(\tau)f(\bar{y})|}{\tau^{1/p+r_i}} d\bar{y} \left. \right\|_{L_{(p)}(E^m)}; \end{aligned} \quad (5)$$

$$\begin{aligned} \|\Delta_s(t)f_3(\bar{x}, t, h)\|_{L_{(p)}(E^m)} &\leq ct \sum_{i=1}^n \left\| \int_{t^{1/\sigma_s}}^h \frac{dv}{v^{1+\lambda_i-\varepsilon+\gamma\sigma_i+\sigma_s-\alpha/pr_n}} \times \right. \\ &\quad \times \int_0^{v^{\sigma_i}} \tau^\gamma d\tau \int_x^{x+v^{\sum_{j=1}^n \sigma_j}} \frac{|\Delta_i(\tau)f(\bar{y})|}{\tau^{1/p+r_i}} d\bar{y} \left. \right\|_{L_{(p)}(E^m)}. \end{aligned} \quad (6)$$

We have

$$\begin{aligned} &\left(\int_0^\infty \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f(\bar{x})\|_{L_{(p)}(E^m)}^p \right)^{1/p} \leq \\ &\leq \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f(\bar{x})\|_{L_{(p)}(E^m)}^p \right)^{1/p} + \left(\int_{h^{\sigma_s}}^\infty \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f(\bar{x})\|_{L_{(p)}(E^m)}^p \right)^{1/p} \leq \\ &\leq \sum_{i=1}^3 \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f_i\|_{L_{(p)}(E^m)}^p \right)^{1/p} + \left(\int_{h^{\sigma_s}}^\infty \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f(\bar{x})\|_{L_{(p)}(E^m)}^p \right)^{1/p} = \\ &= \sum_{i=1}^3 A_i + B. \end{aligned} \quad (7)$$

If we take into account (4), (5), and (6), then after a series of estimates we obtain

$$\begin{aligned} A_1 &= \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t)f_1\|_{L_{(p)}(E^m)}^p \right)^{1/p} \ll ch^{-(\delta+\rho_s\sigma_s)} \|f\|_{L_p^+(E^n)}, \\ A_2, A_3 &\ll ch^{\varepsilon-\rho_s\sigma_s} \|f\|_{L_{p,\alpha}^r(E^n)}. \end{aligned}$$

Consequently,

$$\sum_{i=1}^3 A_i \ll c \left(h^{-(\delta+\rho_s\sigma_s)} \|f\|_{L_p^+(E^n)} + h^{\varepsilon-\rho_s\sigma_s} \|f\|_{L_{p,\alpha}^r(E^n)} \right). \quad (8)$$

It is not difficult to show that B is no greater than the right-hand side of inequality (8); therefore

$$\|f\|_{L_{\vec{p}}(p, p_1, \dots, p_m)(E^m)} \ll c \left(h^{-(\delta + \rho_s \sigma_s)} \|f\|_{L_p(E^n)} + h^{\varepsilon - \rho_s \sigma_s} \|f\|_{L_{p, \alpha}^r(E^n)} \right).$$

Hence, putting $h = 1$, we obtain the required result.

Corollary. If $\varepsilon - \rho_s \sigma_s = 0$, then as $h \rightarrow \infty$

$$\|f\|_{L_{\vec{p}}(p, p_1, \dots, p_m)(E^m)} \ll c \|f\|_{L_{p, \alpha}^r(E^n)}.$$

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CITED LITERATURE

¹ S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950. ² V. P. Il' in, V. A. Solonnikov, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **66**, 205 (1962). ³ L. D. Kudryavtsev, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **55** (1959). ⁴ A. D. Dzhabrailov, Author's abstract of Candidate's dissertation, 1966.

Note: Figure translations are in progress. See original paper for figures.

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