

# ON SECTIONS OVER ZERO-DIMENSIONAL SUBSETS OF QUOTIENT SPACES OF LOCALLY BICOMPACT GROUPS

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1968

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**Abstract**

**Full Text**

UDC 513.831

*MATHEMATICS*

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## ON SECTIONS OVER ZERO-DIMENSIONAL SUBSETS OF QUOTIENT SPACES OF LOCALLY BICOMPACT GROUPS

*(Presented by Academician P. S. Aleksandrov, 13 IV 1967)*

Reiter in <sup>(1)</sup> proved that if in the quotient group  $G/H$  of a locally bicomact group  $G$  with the first axiom of countability one takes an arbitrary derived (i.e., containing no nonempty perfect subset) closed set  $A'$ , then in  $G$  there exists a closed set  $A$  which is mapped onto  $A'$  homeomorphically by means of the natural mapping  $G \rightarrow G/H$ . In the same paper <sup>(1)</sup> Reiter posed the question of extending his result to arbitrary locally bicomact groups. It turns out that such an extension is possible.

**Theorem 1.** *If  $G$  is a locally bicomact group and  $H$  is its closed subgroup, then for every paracompact zero-dimensional\* set  $A' \subseteq G/H = X$  there is a set  $A \subseteq G$  which is mapped onto  $A'$  homeomorphically by means of the natural mapping  $p : G \rightarrow X$ .*

**Remark.** As  $A'$  one may take, for example, any countable subset of  $X$  or any zero-dimensional closed subset of  $X$ .

If in the theorem  $\dim X < \infty$ , then, by Mostert's theorem <sup>(2)</sup> (see also <sup>(3)</sup>), the mapping  $p : G \rightarrow X$  is a locally trivial fibration. In view of the paracompactness and zero-dimensionality of  $A'$ , the mapping  $p : p^{-1}(A') \rightarrow A'$  will already be a trivial fibration, whence the existence of the set  $A$  follows.

In the general case we shall need several lemmas.

**Lemma 1.** *Let  $X$  be the limit of such an inverse spectrum*

$$S = \{X_\alpha, \mathfrak{F}_\alpha^\beta\}, \quad \alpha \in \mathfrak{A},$$

*that: 1) the indices  $\alpha$  are all ordinal numbers less than a certain limit number  $\theta$ ; 2) the projections  $\mathfrak{F}_\alpha^{\alpha+1}$  are locally trivial fibrations; 3) for every limit number  $\beta$ , the element of the spectrum  $X_\beta$  is the limit of the spectrum*

$$S_\beta = \{X_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}, \quad \alpha < \beta.$$

Then for every paracompact zero-dimensional set  $A' \subseteq X_1$  there exists a set  $A$  in  $X = X_0$  which is mapped onto  $A'$  homeomorphically by means of the projection  $\mathfrak{F}_1 : X \rightarrow X_1$ .

**Proof.** The set  $A \subseteq X$  is obtained as the limit of a spectrum  $\{A_\alpha, \mathfrak{F}_\alpha^\beta\}$ ,  $\alpha \in \mathfrak{A}$ , for which  $A_1 = A'$ ,  $A_\alpha \subseteq X_\alpha$ , and the mappings  $\mathfrak{F}_\alpha^\beta : A_\beta \rightarrow A_\alpha$  are homeomorphisms. The sets  $A_\alpha$  are constructed by transfinite induction. (The construction of  $A_{\alpha+1}$  from  $A_\alpha$  uses condition 2) and the fact that the fibration  $\mathfrak{F}_\alpha^{\alpha+1} : (\mathfrak{F}_\alpha^{\alpha+1})^{-1}A_\alpha \rightarrow A_\alpha$  is trivial. The construction of  $A_\beta$  for a limit  $\beta$  uses condition 3).)

**Lemma 2.** If  $G$  is a locally bicompat projective-Lie <sup>(4)</sup> group and  $H$  is its closed subgroup, then the space of the group  $G$

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\* By dimension is meant dimension defined by means of coverings.

is the limit of the spectrum  $S = \{X_\alpha, \delta_\alpha^\beta\}$ ,  $\alpha \in \mathfrak{A}$ , satisfying the conditions of Lemma 1, and  $X_1 = G/H$ .

**Proof.** We construct for  $G$  an analogue of the Pontryagin-Lee series, beginning with the space  $G/H$ . In  $G$  one can <sup>(3)</sup> choose a system of bicompat normal divisors  $G_\alpha$ , indexed by all ordinal numbers  $\alpha$ ,  $2 \leq \alpha \leq \omega_\tau$ , where  $\tau$  is the weight of  $G$  at a point, such that: 1)  $G_{\alpha+1} \subset G_\alpha$ , 2)  $G_\beta = \bigcap_{\alpha < \beta} G_\alpha$  for limit numbers  $\beta$ ; 3) every neighborhood of the identity contains at least one of the normal divisors  $G_\alpha$ , in particular  $\bigcap_\alpha G_\alpha = e$ ; 4) the quotient groups  $G/G_2$  and  $G_\alpha/G_{\alpha+1}$  are Lee groups.

Denote the subgroups  $H \cap G_\alpha$  by  $H_\alpha$ , and  $H$  by  $H_1$ . Since for  $\beta > \alpha$  we have the inclusion  $H_\beta \subseteq H_\alpha$ , the natural projection  $\delta_\alpha^\beta$  of the quotient space  $X_\beta = G/H_\beta$  onto the quotient space  $G/H_\alpha = X_\alpha$  is defined; moreover, for all  $\alpha$  the natural projections  $\delta_\alpha : G \rightarrow X_\alpha$  are defined. Since all subgroups  $H_\alpha$  for  $\alpha \geq 2$  are bicompat, all projections  $\delta_\alpha$  and  $\delta_\alpha^\beta$  for  $\alpha \geq 2$  are perfect.\* Thus the spectrum  $S = \{X_\alpha, \delta_\alpha^\beta\}$ ,  $1 \leq \alpha < \omega_\tau$ , is defined.

Consider a limit number  $\beta \leq \omega_\tau$  (putting  $G_{\omega_\tau} = e$  and  $X_{\omega_\tau} = G$ ). The space  $X_\beta$ , for each  $\alpha$ ,  $1 \leq \alpha < \beta$ , has a continuous mapping  $\delta_\alpha^\beta$  onto the space  $X_\alpha$ , and  $\delta_{\alpha'}^{\alpha'} \cdot \delta_{\alpha'}^\beta = \delta_\alpha^\beta$  for  $\alpha' > \alpha$ , i.e. there is defined a continuous mapping  $f_\beta$  of the space  $X_\beta$  onto the everywhere dense subset  $\bar{X}_\beta$  of the limit of the spectrum  $S_\beta = \{X_\alpha, \delta_\alpha^{\alpha'}\}$ ,  $\alpha < \beta$ , satisfying the relations  $\delta_\alpha^\beta = \pi_\alpha \cdot f_\beta$  (where  $\pi_\alpha$  denotes the projection of the limit of the spectrum  $S_\beta$  onto the element of the spectrum  $X_\alpha$ ). Since the mappings  $\delta_\alpha^\beta$  are bicompat mappings “onto” and  $\delta_{\alpha'}^{\alpha'} \cdot \delta_{\alpha'}^\beta = \delta_\alpha^\beta$  for  $\alpha' > \alpha$ , the mapping  $f_\beta$  is also a bicompat mapping “onto” (if  $\delta_{\alpha'}^{\alpha'}(x_{\alpha'}) = x_\alpha$ , then  $(\delta_{\alpha'}^\beta)^{-1}x_{\alpha'} \subseteq (\delta_\alpha^\beta)^{-1}x_\alpha$ ). Since  $\delta_\alpha^\beta = \pi_\alpha \cdot f_\beta$  and the mapping  $\delta_\alpha^\beta$  is perfect,  $f_\beta$  will also be perfect. Finally, let  $x_1 \neq x_2 \in X_\beta$ , i.e.  $x_1 = g_1H_\beta \neq x_2 = g_2H_\beta$ . If  $\delta_\alpha^\beta(x_1) = \delta_\alpha^\beta(x_2)$ , then  $g_1H_\alpha = g_2H_\alpha$ , whence follows the existence of an index

$\alpha_0 < \beta$  such that  $g_1 H_{\alpha_0} \neq g_2 H_{\alpha_0}$ . (If this were not so, then

$$g_1 H_\beta = g_1 \left( \bigcap_{\alpha < \beta} H_\alpha \right) = \bigcap_{\alpha < \beta} g_1 H_\alpha = \bigcap_{\alpha < \beta} g_2 H_\alpha = g_2 \left( \bigcap_{\alpha < \beta} H_\alpha \right) = g_2 H_\beta,$$

which contradicts the choice of  $x_1$  and  $x_2$ .) Thus there exists an index  $\alpha_0$  for which  $\delta_{\alpha_0}^\beta(x_1) \neq \delta_{\alpha_0}^\beta(x_2)$ . One-to-one-ness, and hence homeomorphism, of  $f_\beta$  is proved.

Finally, let us show that each projection  $\delta_\alpha^{\alpha+1} : X_{\alpha+1} \rightarrow X_\alpha$  is a locally trivial fibration; for this it suffices to show that any quotient group  $H_{\alpha+1}/H_\alpha$  is a manifold ((<sup>3</sup>), Theorem 14).

The quotient group  $H_1/H_2 = H/H \cap G_2$  is homeomorphic to the image of the group  $H$  in the quotient group  $G/G_2$ , i.e. to a closed subgroup of a Lee group, hence to a Lee group. Indeed, let us put in correspondence to the class  $h \cdot H_2 = h(H \cap G_2) \in H_1/H_2$ ,  $h \in H = H_1$ , the class  $f(hH_2) = hG_2 = h \cdot H_2 G_2 \in G/G_2$ . If  $h' \cdot H_2 \neq h'' \cdot H_2$ , then  $h' \cdot G_2 \neq h'' \cdot G_2$ , for otherwise  $h'g' = h''g''$ ,  $g'$  and  $g'' \in G_2$ , i.e.  $(h'')^{-1}h' = g'' \cdot (g')^{-1} \in H \cap G_2 = H_2$ , whence  $h' = h''(g''(g')^{-1}) \in h'' \cdot H_2$ . Thus the mapping  $f$  is one-to-one. It is obviously a mapping "onto." If the set  $F$  is closed in  $H_1/H_2$ , then its inverse image  $\Phi = \Phi \cdot H_2 = \Phi(H \cap G_2)$  is closed in  $H_1$ , and then the set  $\Phi \cdot G_2$  is closed in  $G$  (by virtue of bicomcompactness of  $G_2$ ), i.e. so is its image in  $G/G_2$ , equal to  $f(F)$ . The closedness of  $f$  is proved.

Let us prove the continuity of  $f$ . Let the set  $F'$  be closed in the image of the subgroup  $H$  in the quotient group  $G/G_2$ . Then the inverse image  $\tilde{\Phi}$  of the set is closed in  $G$

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\* That is, they are closed (the image of a closed set is closed) and bicomcompact (the inverse image of a point is bicomcompact) mappings.

$F'$ , and hence the set  $\Phi' = \tilde{\Phi} \cap H$  is closed in  $H$ . The set  $\Phi'$  coincides with the set  $\Phi' \cdot H_2$ , since

$$\Phi' = \tilde{\Phi} \cap H \subseteq (\tilde{\Phi} \cap H) \cdot H_2 = \Phi' \cdot H_2 = (\tilde{\Phi} \cap H) \cdot (G_2 \cap H) \subseteq (\tilde{\Phi} \cdot G_2) \cap H = \tilde{\Phi} \cap H = \Phi'.$$

The image of the set  $\Phi'$  in  $H_1/H_2$  is closed and, by the one-to-one character of  $f$ , coincides with the set  $f^{-1}(F')$ . The homeomorphism  $f$  is proved.

The space of the factor group  $H_\alpha/H_{\alpha+1}$  for  $\alpha > 1$  is bicomcompact and homeomorphic to the image of the group  $H_\alpha$  in the Lie group  $G_\alpha/G_{\alpha+1}$ , i.e. the spaces  $H_\alpha/H_{\alpha+1}$  for  $\alpha > 1$  are also manifolds. Thus the local triviality of the decompositions

$$\omega_\alpha^{\alpha+1} : X_{\alpha+1} \rightarrow X_\alpha$$

in the case of a projective-Lie group  $G$  is proved, i.e. the lemma is also proved.

From Lemmas 1 and 2 the validity of the theorem for projective-Lie groups follows immediately.

If now the group  $G$  is arbitrary, then it always contains an open projective-Lie subgroup  $\Gamma$  <sup>(5)</sup>, and the factor space  $X = G/H$  decomposes into a sum of pairwise disjoint open sets  $X_\nu$ , whose inverse images  $p^{-1}(X_\nu)$  in  $G$  under the natural mapping  $p : G \rightarrow X$  have, for some  $a_\nu \in G$ , the form  $\Gamma \cdot a_\nu \cdot H$  (see, for example, <sup>(3)</sup>, Lemma 1). The mapping  $p$  of the set  $\Gamma \cdot a_\nu \subseteq \Gamma \cdot a_\nu \cdot H$  onto  $X_\nu$  can be identified <sup>(3)</sup>, Lemma 1, with the natural mapping of the projective-Lie group  $\Gamma$  onto the factor space

$$\Gamma/(\Gamma \cap a_\nu \cdot H \cdot a_\nu^{-1}),$$

i.e. everything has been reduced to the case of a projective-Lie group. The theorem is completely proved.

In the case of countable sets, Theorem 1 can be proved under more general assumptions. (All spaces considered are assumed to be completely regular.)

**Definition.** A space  $X$  has the almost everywhere first axiom of countability if  $X$  contains an everywhere dense subset of points of countable character in  $X$ .

**Theorem 2.** *Let an open mapping  $f : X \rightarrow Y$  be given such that every set  $f^{-1}(y)$ ,  $y \in Y$ , has the almost everywhere first axiom of countability. Then for every  $\sigma$ -discrete (in itself) paracompact set  $N \subseteq Y$  there is a set  $M \subseteq X$ , mapped onto  $N$  by  $f$  homeomorphically (in short, in  $X$  over  $N$  there exists a section).*

**Corollary 1.** *If an open mapping  $f : X \rightarrow Y$  of a paracompact <sup>(6)</sup> space  $X$  with the first axiom of countability (in particular, complete with the first axiom of countability or perfectly mapped onto a metrizable space with the first axiom of countability) is given, then for every  $\sigma$ -discrete (in itself) paracompact (for example, countable) set  $N \subseteq Y$ , in  $X$  over  $N$  there exists a section.*

It is useful to compare this corollary with Theorem 1 from <sup>(1)</sup>.

**Corollary 2.** *If  $G$  is a topological group,  $H'$  and  $H''$  are its closed subgroups,  $H' \supseteq H''$ , and the factor spaces  $X' = G/H'$  and  $X'' = G/H''$  are metrizable, then in  $X''$ , over any  $\sigma$ -discrete set  $N \subseteq X'$ , with respect to the natural projection  $p : X'' \rightarrow X'$ , there exists a section.*

**Corollary 3.** *If a group  $G$  is almost metrizable <sup>(7)</sup>, and  $H'$  and  $H''$  are its closed subgroups,  $H' \supseteq H''$ , and the factor space  $H'/H''$  is metrizable, then with respect to the natural mapping  $p : G/H'' \rightarrow G/H'$ , for every  $\sigma$ -discrete set closed in  $G/H'$  there exists a section in  $G/H''$ .*

**Theorem 3.** *If a group  $G$  is almost metrizable and  $H$  is its closed subgroup, then with respect to the natural projection  $p : G \rightarrow G/H$ , for every  $\sigma$ -discrete set  $N$  closed in  $G/H$ , there exists a section in  $G$ .*

Instead of closedness of  $N$  one may require that  $N$  be a paracompact subset of type  $G_\delta$  in  $G/H$ .

The proof of Theorem 3 is analogous to the proof of Theorem 1.

From Theorem 3 follows the validity of Theorem 1, however, only for a  $\sigma$ -discrete set  $A'$  closed in  $X$ .

Let us also note that spaces  $X$  of point-countable type, all points of which are sets of type  $G_\delta$  in  $X$ , satisfy the first axiom of countability; whence it follows

**Theorem 4.** *The quotient space  $G/H$  of an almost metrizable group  $G$  is metrizable if and only if at least one of its points has type  $G_\delta$  in  $G/H$ .*

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Received  
22 III 1967

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*Note: Figure translations are in progress. See original paper for figures.*

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