

ON SOLUTIONS OF FINITE FORM OF CERTAIN HILL EQUATIONS

L. S. Yakupov

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Abstract

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MATHEMATICS

L. S. Yakupov

ON SOLUTIONS OF FINITE FORM OF CERTAIN HILL EQUATIONS

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It is known that the radial wave equation reduces to the form

$$d^2y/d\tau^2 + (\delta + \varepsilon e^\tau + \mu e^{2\tau})y = 0 \quad (1)$$

(the Hill equation with imaginary period), for which, under certain relations among δ, ε, μ , solutions in closed form have been found ¹.

Solutions of finite form can also be found for the Hill equation with real period, analogous to (1):

$$d^2y/d\tau^2 + (\delta + \varepsilon_c \cos \tau + \varepsilon_s \sin \tau + \mu \cos 2\tau)y = 0. \quad (2)$$

Equation (2), under certain relations among $\delta, \varepsilon_c, \varepsilon_s, \mu$, has solutions in the form of a segment of a Fourier series multiplied by

$$E(\tau) = \exp \left[\int (\lambda_0 + \lambda \sin \tau) d\tau \right]. \quad (3)$$

For $y = E(\tau)f(\tau)$, equation (2) is satisfied if

$$f'' + 2(\lambda_0 + \lambda \sin \tau)f' = (\alpha + 2\beta \cos \tau)f, \quad (4)$$

$$\delta = -(\alpha + \lambda_0^2 + \mu), \quad \varepsilon_c = -(2\beta + \lambda), \quad \varepsilon_s = -2\lambda_0\lambda, \quad \mu = \lambda^2/2.$$

From (4), after the substitution

$$f(\tau) = \sum_{n=0}^N \frac{a_n \cos n\tau + b_n \sin n\tau}{c_n \cos(n + \frac{1}{2})\tau + d_n \sin(n + \frac{1}{2})\tau}$$

and elementary transformations, one obtains a system of algebraic equations, the study of which leads to Theorems 1 and 2.

Theorem 1. Let λ_0 be a constant, $\lambda = \sqrt{2\mu}$, $N = 0, 1, \dots$, $L_1 = \text{diag}\{2\lambda_0, 4\lambda_0, \dots, 2N\lambda_0\}$ be a diagonal matrix; B, A be codiagonal matrices:

$$B = \text{codiag} \left\{ \begin{array}{cccccc} 1, & \frac{(N+2)\lambda}{2^2}, & \frac{(N+3)\lambda}{3^2}, & \dots & \dots, & \frac{2N\lambda}{N^2} \\ (N-1)\lambda, & (N-2)\lambda, & \dots & \dots, & \lambda & \end{array} \right\},$$

$$A = \text{codiag} \left\{ \begin{array}{cccccc} 0, & \frac{(N+1)\lambda}{1}, & \frac{(N+2)\lambda}{2^2}, & \dots & \dots, & \frac{2N\lambda}{N^2} \\ 2N\lambda, & (N-1)\lambda, & (N-2)\lambda, & \dots & \lambda & \end{array} \right\}.$$

If $\varepsilon_c = -(2N+1)\lambda$, $\varepsilon_s = -2\lambda_0\lambda$, and the sum $\delta + \mu + \lambda_0^2$ is equal to an eigenvalue of the matrix

$$A^* = \left[\begin{array}{c|cc} A & 0 & \dots 0 \\ \hline 0 & & -L_1 \\ \vdots & L_1 & \\ 0 & & B \end{array} \right]$$

and to this value there corresponds the eigenvector $(a_0^*, \dots, a_N^*, b_1^*, \dots, b_N^*)$, then equation (2) has the solution

$$y = E(t) \sum_{n=0}^N (a_n^* \cos nt + b_n^* \sin nt).$$

Corollary 1.1. If $\varepsilon = -(2N+1)\lambda$, the sum $\delta + \mu$ is equal to an eigenvalue of the matrix A , and to this value there corresponds the eigenvector $(a_0, \dots, a_N)$, then the equation

$$d^2y/d\tau^2 + (\delta + \varepsilon \cos \tau + \mu \cos 2\tau)y = 0 \quad (5)$$

has a 2π -periodic even solution

$$y = e^{-\lambda \cos \tau} \sum_{n=0}^N a_n \cos n\tau; \quad (6)$$

if, however, $\delta + \mu$ is equal to an eigenvalue of the matrix B , to which the vector $(b_1, \dots, b_N)$ corresponds, then equation (5) has a 2π -periodic odd solution

$$y = e^{-\lambda \cos \tau} \sum_{n=1}^N b_n \sin n\tau. \quad (7)$$

Corollary 1.2. The solutions (6) and (7), as $N \rightarrow \infty$, tend to 2π -periodic Mathieu functions of integral order.

Theorem 2. Let $L_2 = \text{diag}\{\lambda_0, 3\lambda_0, \dots, (2N+1)\lambda\}$,

$$C = \text{codiag} \left\{ \begin{array}{cccccc} (N+2)\lambda, & (N+3)\lambda, & \dots, & \dots, & \dots, & (2N+1)\lambda \\ (1/2)^2 + (N+1)\lambda, & (3/2)^2, & (5/2)^2, & \dots, & (N-1/2)^2, & (N+1/2)^2 \\ N\lambda, & (N-1)\lambda, & \dots, & \dots, & \lambda & \end{array} \right\};$$

D is the codiagonal matrix obtained from C by changing the sign of λ in the element c_{11} .

If $\varepsilon_c = -2(N+1)\lambda$, $\varepsilon_s = -2\lambda_0\lambda$, and the sum $\delta + \mu + \lambda_0^2$ is equal to an eigenvalue of the matrix

$$C^* = \begin{bmatrix} C & -L_2 \\ L_2 & D \end{bmatrix},$$

to which there corresponds the eigenvector $(c_0^*, \dots, c_N^*, d_0^*, \dots, d_N^*)$, then equation (2) has the solution

$$y = E(\tau) \sum_{n=0}^N [c_n^* \cos(n+1/2)\tau + d_n^* \sin(n+1/2)\tau].$$

Corollary 2.1. If $\varepsilon = -2(N+1)\lambda$, $\delta + \mu$ is equal to an eigenvalue of the matrix C , and to this value there corresponds the eigenvector $(c_0, \dots, c_N)$, then equation (5) has a 4π -periodic even solution

$$y = e^{-\lambda \cos \tau} \sum_{n=0}^N c_n \cos(n+1/2)\tau; \quad (8)$$

if, however, $\delta + \mu$ is equal to an eigenvalue of the matrix D , to which there corresponds the eigenvector $(d_0, \dots, d_N)$, then equation (5) has a 4π -periodic odd solution

$$y = e^{-\lambda \cos \tau} \sum_{n=0}^N d_n \sin(n+1/2)\tau. \quad (9)$$

Corollary 2.2. The solutions (8) and (9), as $N \rightarrow \infty$, tend to 4π -periodic Mathieu functions of integral order.

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Note: Figure translations are in progress. See original paper for figures.

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