

ON INVERSE BOUNDARY-VALUE PROBLEMS WITH UNKNOWN COEFFICIENTS FOR SOME QUASILINEAR EQUATIONS

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Abstract

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MATHEMATICS

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ON INVERSE BOUNDARY-VALUE PROBLEMS WITH UNKNOWN COEFFICIENTS FOR SOME QUASILINEAR EQUATIONS

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In the present note we consider inverse boundary-value problems with two unknown coefficients for an equation of parabolic type and for an ordinary differential equation with a quasilinear right-hand side. Questions of existence, uniqueness, and stability of the solution are considered, and the convergence of the proposed algorithms is investigated; some results from ⁽¹⁻³⁾ are generalized.

I. Problem for an equation of parabolic type.

Suppose it is required to find the triple of functions $\{a(x, t), c(t), u(x, t)\}$ from the conditions

$$\begin{aligned} a(x, t)b(x, t)u_{xx} - c(t)d(x, t)u - u_t &= H(x, t, a, c, u), \quad 0 < x < l, \\ 0 < t &\leq T; \end{aligned} \quad (1)$$

$$u(0, t) = f_0(t), \quad u(l, t) = f_n(t), \quad 0 \leq t \leq T; \quad (2_1)$$

$$\begin{aligned} u(l_k - 0, t) &= u(l_k + 0, t), \quad a(l_k - 0, t)b(l_k - 0, t)u_x(l_k - 0, t) = \\ &= a(l_k + 0, t)b(l_k + 0, t)u_x(l_k + 0, t), \end{aligned}$$

$$0 = l_0 < l_1 < \dots < l_{n-1} = l_n = l, \quad k = 1, 2, \dots, n-1, \quad 0 \leq t \leq T;$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l, \quad \varphi(0) = f_0(0), \quad \varphi(l) = f_n(0); \quad (3)$$

$$u(l_k, t) = f_k(t), \quad 0 \leq t \leq T, \quad \varphi(l_k) = f_k(0), \quad k = 1, 2, \dots, n-1; \quad (4)$$

$$-a(0, t)u_x(0, t) = g(t), \quad 0 \leq t \leq T; \quad (5)$$

$$-a(l, t)u_x(l, t) = c(t)\psi(t) + g_1(t), \quad 0 \leq t \leq T, \quad (6)$$

where $b(x, t) > 0$ and $d(x, t)$ are known functions, piecewise constant in x for $0 \leq x \leq l$, with discontinuities only at the points $x = l_k$, $k = 1, 2, \dots, n-1$, and continuous in t for $0 \leq t \leq T$; $H(x, t, a, c, u)$ is a continuous function of its arguments in $\Pi_k\{l_{k-1} \leq x \leq l_k, 0 \leq t \leq T, 0 \leq a \leq A, |c| \leq C, |u| \leq Q\}$, $k = 1, 2, \dots, n$; $\varphi(x)$ is a continuous function for $0 \leq x \leq l$, while the functions $f_k(t)$, $k = 0, 1, 2, \dots, n$, $g(t)$, $\psi(t)$, $g_1(t)$ are continuous for $0 \leq t \leq T$.

Definition 1. The triple of functions $\{a(x, t), c(t), u(x, t)\}$ will be called a solution of problem (1)–(6) if these functions satisfy the following requirements: 1) $a(x, t) > 0$ is a continuous function of t for $0 \leq t \leq T$, piecewise constant in x for $0 \leq x \leq l$, having discontinuities only at $x = l_k$, $k = 1, 2, \dots, n-1$; 2) $c(t)$ is a continuous function of t for $0 \leq t \leq T$; 3) $u(x, t)$ is continuous in $\bar{\Omega}\{0 \leq x \leq l, 0 \leq t \leq T\}$; $u_x(x, t)$, $u_{xx}(x, t)$, $u_t(x, t)$ are defined and continuous in $\Omega_k\{l_{k-1} < x < l_k, 0 < t \leq T\}$, $k = 1, 2, \dots, n$, and the limits $u_x(0+0, t)$, $u_x(l_k+0, t)$, $u_x(l_k-0, t)$, $k = 1, 2, \dots, n-1$, $u_x(l-0, t)$ exist; 4) all relations (1)–(6) are fulfilled.

Let $a_k(t)$, $b_k(t)$, $d_k(t)$, and $u_k(x, t)$ be respectively the values of $a(x, t)$, $b(x, t)$, $d(x, t)$, and $u(x, t)$ in the regions $\Omega_k\{l_{k-1} \leq x \leq l_k, 0 \leq t \leq T\}$.

Theorem 1. Let, in problem (1)–(6), $n = 1$, and let the following conditions be satisfied:

- a) $0 < f_{\min} \leq f_0(t) \leq f_{\max}$, $f_1(t) \equiv 0$, $\varphi(x) \geq 0$, $\varphi_x(x) \leq 0$, $\varphi_x(0) < 0$, $\varphi_{xx}(x) \geq 0$, $g_{\max} \geq g(t) \geq g_{\min} > 0$, $\psi_{\max} \geq \psi(t) \geq \psi_{\min} > 0$, $-g_{1\max} \leq g_1(t) \leq 0$, $0 \leq d_1(t) \leq d_{1\max}$, $0 < b_{\min} \leq b_1(t) \leq b_{\max}$, where $f_{\min}$, $f_{\max}$, $g_{\min}$, $g_{\max}$, $\psi_{\min}$, $\psi_{\max}$, $g_{1\max}$, $d_{\max}$, $b_{\min}$, $b_{\max}$ are certain positive constants;
- b) $H \leq 0$, $H_{xx} \leq 0$, $H_u \leq 0$, $H_{uu} \leq 0$, $H_{xu} \geq 0$, $0 \leq H(0, t, a, c, f_0) + f_{0t}(t)$, $H(l, t, a, c, 0) \equiv 0$, $|H(x, t, a, c, u)| \leq E + F|u|$, where $E = \text{const} > 0$, $F = \text{const} > 0$;
- c) $a_1(0)b(0)\varphi_{xx}(0) - c(0)d(0)\varphi(0) + f_{0t}(0) = H(0, 0, a(0), c(0), f_0(0))$, $\varphi_{xx}(l) = 0$, where $a_1(0) = g(0)[- \varphi_x(0)]^{-1}$, $c(0) = [\psi(0)]^{-1}\{a_1(0)\varphi_x(l) - g_1(0)\}$;
- d) $\sqrt{T}[I_1\sqrt{I_3} + \sqrt{T}I_2 + \sqrt{I_3}] \leq 1$, where

$$I_1 = \|\varphi_{xxx}\|_0 + T\{\|H_{xxx}\|_0 + \|H_{xu}\|_0 f_{\max} l_1^{-1} + \|H_{uu}\|_0 f_{\max}^2 l_1^{-2} + (ab_{\min})^{-1}[\|H\|_0 + \|f_{0t}(t)\|_0 + Cd_{\max} f_{\max}]\},$$

$$I_2 = 2\|H_{xu}\|_0 + \|H_u\|_0 + 2\|H_{uu}\|_0 f_{\max}, \quad I_3 = \|H_{uu}\|_0 l_1^2;$$

$\|\cdot\|_0$ denotes the maximum of the modulus of the given quantity in the domain under consideration (the norm in the uniform metric);

e) $H, H_x, H_{xx}, H_{xxx}, H_{xxxx}, H_{xuu}, H_{xuuu}, H_{xxu}, H_{xxuu}, H_{xxxu}, H_{xu}, H_{uuuu}, H_{uuu}, H_{uu}, H_u, H_t, H_c, H_a$ are continuous functions of their arguments in

$$\bar{\Pi}_1 \{0 \leq x \leq l_1, 0 \leq t \leq T, 0 < \alpha \leq a \leq A, |c| \leq C, |u| \leq Q\},$$

where

$$A = g_{\max} f_{\min}^{-1} l_1, \quad C = A f_{\max} l_1^{-1} \psi_{\min}^{-1} + g_{1 \max} \psi_{\min}^{-1},$$

$$\alpha = g_{\min} [f_{\max} + 2^{-1} l_1^2 (T I_3)^{-1/2}]^{-1} l_1, \quad Q = [FT + f_{\max}] \exp\{ET\},$$

and $\varphi(x), \varphi_x(x), \varphi_{xx}(x), \varphi_{xxx}(x), \varphi_{xxxx}(x), f_0(t), f_{0t}(t), f_{0tt}(t), g(t), g_t(t), \psi(t), \psi_t(t), g_1(t), g_{1t}(t), b_1(t), b_{1t}(t), d_1(t), d_{1t}(t)$ are continuous and bounded functions of their arguments in the domains of definition, respectively, of $\varphi(x), f_0(t), g(t), \psi(t), g_1(t), b_1(t), d_1(t)$.

Then a solution of problem (1)–(6) exists.

Remark. In this theorem, by assuming the appropriate smoothness and compatibility conditions for the data of the problem to be satisfied, one can obtain arbitrary smoothness of $a_1(t), c(t), u_1(x, t)$.

The proof of the theorem is carried out by the method of successive approximations. We indicate the scheme of the method of successive approximations for solving problem (1)–(6) for arbitrary $n \geq 1$:

$$a_k^{(s)}(t) b_k(t) u_{kxx}^{(s)} - c^{(s)}(t) d_k(t) u_k^{(s)} - u_{kt}^{(s)} = H(x, t, a^{(s)}, c^{(s)}, u^{(s)}),$$

$$l_{k-1} < x < l_k, \quad 0 < t \leq T; \quad (7)$$

$$u_k^{(s)}(l_{k-1}, t) = f_{k-1}(t), \quad u_k^{(s)}(l_k, t) = f_k(t), \quad 0 \leq t \leq T; \quad (8)$$

$$u_k^{(s)}(x, 0) = \varphi(x), \quad \varphi(l_{k-1}) = f_{k-1}(0), \quad \varphi(l_k) = f_k(0), \quad l_{k-1} \leq x \leq l_k, \\ k = 1, 2, \dots, n-1; \quad (9)$$

$$-a_1^{(s+1)}(t) u_{1x}^{(s)}(0, t) = g(t), \quad 0 \leq t \leq T; \quad (10)$$

$$a_{k+1}^{(s+1)}(t) b_{k+1}(t) u_{k+1,x}^{(s)}(l_k, t) = a_k^{(s+1)}(t) b_k(t) u_{kx}^{(s)}(l_k, t), \quad 0 \leq t \leq T; \quad (11)$$

$$-a_n^{(s+1)}(t) u_{nx}^{(s)}(l, t) = c^{(s+1)}(t) \psi(t) + g_1(t), \quad 0 \leq t \leq T. \quad (12)$$

It is clear that, for the unrestricted feasibility of this iterative process, one must have

$$g(t) u_{1x}^{(s)}(0, t) < 0, \quad u_{kx}^{(s)}(l_k, t) u_{k+1,x}^{(s)}(l_k, t) > 0, \\ s = 0, 1, 2, \dots, \quad \psi(t) \neq 0, \quad 0 \leq t \leq T. \quad (13)$$

Definition 2. We shall say that a solution of problem (1)–(6) belongs to the class $C_{0,0,21}(\bar{\Pi}_k)$ if $a_k(t) \in C[0, T], c(t) \in C[0, T], u_k(x, t) \in C_{21}(\bar{\Omega}_k), k = 1, 2, \dots, n$.

Theorem 2. Let a solution of problem (1)–(6) exist and belong to the class $C_{0,0,21}(\overline{\Pi}_k)$. If, for all $s = 0, 1, 2, \dots$, the solutions of problem (7)–(9) from $C_{2,1}(\overline{\Omega}_h)$, $u_k^{(s)}(t)$, $k = 1, 2, \dots, n$, $c^{(s)}(t)$, are uniformly bounded with respect to s for $0 \leq t \leq T$, and the following conditions are satisfied:

a)

$$g(t)u_{1x}^{(s)}(0, t) < 0, \quad u_{kx}^{(s)}(l_k, t)u_{k+1,x}^{(s)}(l_k, t) > 0, \quad k = 1, 2, \dots, n-1,$$

$$u_{k+1,x}(l_k, t) \neq 0, \quad k = 0, 1, 2, \dots, n-1, \quad \psi(t) \neq 0;$$

b) $H, H_a, H_c, H_t, H_x, H_u, H_{xu}, H_{tu}, H_{uu}$ are continuous functions of their arguments in $\overline{\Pi}_k$; $\varphi(x), \varphi_x(x), \varphi_{xx}(x)$ are continuous in $\overline{D}_k\{l_{k-1} \leq x \leq l_k\}$. $d_k(t), b_k(t), f_k(t), g(t), \psi(t), g_1(t), f_{kt}(t), b_{kt}(t), d_{kt}(t)$ are continuous functions of t for $0 \leq t \leq T$.

Then the triple of functions $\{a_k^{(s)}(t), c^{(s)}(t), u_k^{(s)}(x, t)\}$, obtained by the method of successive approximations according to the scheme (7)–(12), as $s \rightarrow +\infty$, converges uniformly to the solution of problem (1)–(6) with rate $M_1 H_2^s [s!]^{-1/2}$. Here $M_1 > 0$, $M_2 > 0$ are certain positive constants depending on the data of the problem.

The solution of (1)–(6) can also be found by a difference-iterative method, which consists in first constructing the solution of a difference approximating problem for (1)–(6), and then using this latter solution to find the solution of the differential problem as the mesh sizes of the difference grid tend to zero. It turns out that the method of successive approximations for finding the solution of the difference problem under conditions analogous to those of Theorem 2 converges at the rate of a geometric progression. It is proved that the solution of problem (1)–(6) is unique and stable.

II. Problem for an ordinary differential equation

Suppose it is required to find $a(x)$, c , $u(x)$ from the conditions:

$$a(x)b(x)u_{xx} - cd(x)u = H(x, a, c, u), \quad 0 < x < l; \quad (14)$$

$$u(0) = f_0, \quad u(l) = f_n; \quad (15_1)$$

$$u(l_k - 0) = u(l_k + 0), \quad a(l_k - 0)b(l_k - 0)u_x(l_k - 0) =$$

$$= a(l_k + 0)b(l_k + 0)u_x(l_k + 0);$$

$$0 = l_0 < l_1 \dots < l_{n-1} < l_n = l, \quad k = 1, 2, \dots, n-1; \quad (15_2)$$

$$u(l_k) = f_k, \quad k = 1, 2, \dots, n-1; \quad (16)$$

$$-a(0)u_x(0) = g; \quad (17)$$

$$-a(l)u_x(l) = c\psi + g_1, \quad (18)$$

where $b(x) > 0$, $d(x) > 0$ are known piecewise-constant functions for $0 \leq x \leq l$ with discontinuities only at the points $x = l_k$; $H(x, a, c, u)$ is a continuous function of its arguments in $\overline{\Pi}_k\{l_{k-1} \leq x \leq l_k, 0 \leq a \leq A, 0 \leq c \leq C, |u| \leq Q\}$, and $f_k, k = 0, 1, 2, \dots, n, g, \psi, g_1$ are given constants.

Definition 3. A triple of quantities $\{a(x), c, u(x)\}$ will be called a **solution** of problem (14)–(18) if these functions satisfy the following requirements: 1) $a(x) > 0$ is a piecewise-constant function for $0 \leq x \leq l$, having discontinuities only at the points $x = l_k$; 2) $c > 0$ is some constant; 3) $u(x)$ is continuous in $\overline{D}\{0 \leq x \leq l\}$; $u_x(x), u_{xx}(x)$ are defined and continuous in $\overline{D}_k\{l_{k-1} < x < l_k\}$, $k = 1, 2, \dots, n$, and the limits $u_x(0+0), u_x(l_k+0), u_x(l_k-0), k = 1, 2, \dots, n, u_x(l-0)$ exist; 4) all relations (14)–(18) are satisfied.

The solution of problem (14)–(18) is sought by a difference-iterative method. First, by the method of successive approximations according to a scheme analogous to

(7)–(12), the solution of the approximating difference problem for (14)–(18) is constructed, and then this latter solution is used to find the solution of the differential problem as the steps of the difference mesh tend to zero.

Theorem 3. Suppose that the following conditions are satisfied:

a)

$$f_{k-1} > f_k \operatorname{ch} \left[\sqrt{\gamma \chi_k d_k b_k^{-1} \psi^{-1} (l_k - l_{k-1})} \right] > 0, \quad k = 1, 2, \dots, n, \quad g \geq 0,$$

$$\psi > 0, \quad g_1 = 0, \quad b_k > 0, \quad d_k > 0, \quad \max |H(x, a, c, 0)| \leq c_0 d_k f_k, \quad k = 1, 2, \dots, n,$$

$$2\partial e x_n = (f_{n-1} - f_n)(l_n - l_{n-1}), \quad \chi_k = \chi_{k+1} b_k b_{k+1}^{-1} (f_{k-1} - f_k) f_k - f_{k+1})^{-1} \times \\ \times (l_{k+1} - l_k)(l_k - l_{k-1})^{-1}, \quad c_0 = q_n \psi^{-1} a_n,$$

$$q_k = \frac{\sqrt{\chi_k d_k b_k^{-1} \psi^{-1}}}{\operatorname{sh} \left[\sqrt{\chi_k d_k b_k^{-1} \psi^{-1} (l_k - l_{k-1})} \right]} \left\{ f_{k-1} - f_k \operatorname{ch} \left[\sqrt{\chi_k d_k b_k^{-1} \psi^{-1} (l_k - l_{k-1})} \right] \right\} (1 - \varepsilon),$$

$$0 < \varepsilon < 1, \quad a_1 = g p_1^{-1}, \quad a_{k+1} = a_k d_k b_k^{-1} p_{k+1}^{-1} q_k,$$

$$p_k = (l_k - l_{k-1})^{-1} \{ f_{k-1} - f_k + 2^{-1} (l_k - l_{k-1})^2 [d_k b_k^{-1} \chi_k \psi^{-1} (f_{k-1} + f_k) + \\ + (8Ab_k)^{-1} \delta_{xx} H \|_0 (l_k - l_{k-1})^2] \}, \quad A = \operatorname{const} > 0.$$

b) $H \leq 0$, $H_u \leq 0$, H_x are continuous and bounded functions of their arguments.

Then the solution of problem (14)–(18) exists, and the difference-iteration algorithm for solving the problem converges.

Under appropriate conditions, a theorem on the uniqueness and stability of the solution of problem (14)–(18) is proved.

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Note: Figure translations are in progress. See original paper for figures.

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