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P. A. Shvartsman

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Abstract

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MATHEMATICS

P. A. SHVARTSMAN

ON THE ANALYTIC PROPERTIES OF A POSITIVELY DIRECTED SYMPLECTIC CURVE NEAR A BRANCH POINT OF ITS SPECTRUM

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1. We consider a differential system with periodic coefficients:

$$dX/dt = JH(t)X, \quad H(t+T) = H(t), \quad (1)$$

where $H(t)$ is a real symmetric matrix of order $n = 2m$, positive definite for all values of the argument, and the matrix J has the form

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where I_m is the identity matrix of order m .

By $X(t)$ we shall denote the matrizant of system (1), i.e., the matrix solution of equation (1) satisfying the initial condition $X(0) = I$. As is known ⁽¹⁾, $X(t)$ is a symplectic matrix for all values of the argument, i.e., the identity $X^* J X = J$ holds.

The matrizant of an equation of the form (1) with a positive definite coefficient matrix $H(t)$ is called a positively directed symplectic curve.

Suppose that, for some value t_0 of the argument, the matrix $X(t_0)$ has a multiple eigenvalue ρ_0 lying on the unit circle $|\rho_0| = 1$, and that the matrix-function $H(t)$, or, what is the same, $X(t)$, is analytic in a neighborhood of the point t_0 . It is assumed that in the neighborhood under consideration of the point t_0 (for $t \neq t_0$) all values of the matrizant $X(t)$ are matrices of simple type.

Denote by $D(\rho; t)$ the characteristic polynomial of the matrix $X(t)$. The equation $D(\rho; t) = 0$ defines ρ as a multivalued analytic function of the argument t . Let $\rho_j(t)$, $j = 1, \dots, d$, be those multivalued branches of $\rho(t)$ which at $t = t_0$ take the value ρ_0 :

$$\rho_j(t) = \rho_0 + \sum_{k=1}^{\infty} c_{jk}(t - t_0)^{k/l_j} \quad (j = 1, \dots, d). \quad (2)$$

Theorem 1. 1°. The number d of multivalued branches of the analytic function $\rho(t)$ of the form (2) coincides with the proper multiplicity of the root ρ_0 of the equation $D(\rho; t_0) = 0$.

2°. The numbers $l_1, \dots, l_d$ coincide with the orders of the elementary divisors of the matrix $X(t_0)$ corresponding to its eigenvalue ρ_0 .

3°. The coefficients c_{j1} ($j = 1, \dots, d$) in the expansions (2) are nonzero and have the form $c_{j1} = \sigma_{j1}^{1/l_j}$, where the numbers σ_{j1} are real, and the sign of the number σ_{j1} coincides with the sign of the corresponding sign-definite (see (4)) elementary divisor.

Theorem 1 is a generalization of a theorem of M. G. Krein and G. Ya. Lyubarskii (2), proved under the assumption that the symplectic

...curve is the monodromy matrix of a certain canonical differential equation of positive type.

The proof of Theorem 1 is based on a theorem* of V. B. Lidskii (3). From the expansions (2) there follows a qualitative picture of the behavior of the eigenvalues (multipliers) of the matrixant $X(t)$ near the point t_0 .

To a sign-definite elementary divisor

$$\pm(\rho - \rho_0)^{q_j} \quad (3)$$

there correspond q_j multipliers of the form (2) of the matrix $X(t)$, which for $t = t_0$ coalesce into ρ_0 .

Let q_j increase continuously from $t_0 - \varepsilon$ to $t_0 + \varepsilon$; then, when q_j is odd, exactly one of the multipliers under consideration moves all the time along the unit circle in the direction corresponding to its kind (4). This is a multiplier of the first kind if the elementary divisor (3) is positive, and of the second kind otherwise. The remaining multipliers corresponding to (2) move outside the unit circle, pairwise symmetrically with respect to it. In particular, if the matrix $X(t_0)$ is reducible to diagonal form, then all multipliers move before and after the collision along the unit circle in the direction corresponding to their kind.

If q_j is even and (3) is positive, then before the collision exactly two multipliers among (2) move along the unit circle toward one another, and after the collision not a single multiplier among (2) remains on the unit circle. If, however, (3) is negative, then before the collision all multipliers move outside the unit circle, and after the collision exactly two of them move along the unit circle in opposite directions.

This picture can be supplemented in the case $\rho_0 = \pm 1$. For example, if (3) is positive and $q_j \equiv 0 \pmod{4}$, then before the collision exactly two multipliers (2) move toward one another along the unit circle and exactly two move toward one another along the real axis. After the collision neither on the circle nor on the axis does any multiplier among (2) remain. If q_j is even, but $q_j \not\equiv 0 \pmod{4}$, and (3) is still positive, then before the collision there are exactly two multipliers on the circle and none on the real axis, while after the collision there are exactly two on the real axis and none on the unit circle, etc.

2. To the multivalued analytic functions (2) there correspond multivalued analytic vector-functions

$$\xi_j(t) = \sum_{k=0}^{\infty} \chi_{kj}(t-t_0)^{k/l_j} \quad (j = 1, \dots, d) \quad (4)$$

(χ_{kj} are n -dimensional vectors)—eigenvectors of the matrixant corresponding to the eigenvalues (2).**

We introduce the indefinite scalar product $[x, y] = -i(Jx, y)$.

Theorem 2. For the vectors $\chi_{1j}, \chi_{2j}, \dots$ ($j = 1, \dots, d$) occurring in the expansions (4), the following relations hold:

$$[\chi_{pj}, \chi_{qj}] = 0 \quad \text{for} \quad p + q < l_j - 1.$$

For $p + q = l_j - 1$,

$$[\chi_{pj}, \chi_{qj}] = (H_0 \chi_0, \chi_0) / |c_{1j}| \neq 0 \quad (H_0 = H(t_0))$$

independently of the choice of p and q .

* The author expresses deep gratitude to V. B. Lidskii, who provided the opportunity to become acquainted with his work ⁽³⁾ in manuscript before its publication.

** Recall that, by assumption, t_0 is an isolated point, so that in a neighborhood of t_0 ($t \neq t_0$) the matrix $X(t)$ is of simple type and has n eigenvectors.

Theorem 3. Let

$$\xi(t) = \sum_{k=0}^{\infty} \chi_k(t-t_0)^{k/l}, \quad \eta(t) = \sum_{j=0}^{\infty} \xi_j(t-t_0)^{j/m},$$

be two distinct multivalued vector-functions from among (4). Then from $[\chi_k, \xi_j] \neq 0$ it follows that $j/l + k/m \geq 1$. In particular, for $l = m$, $[\chi_k, \xi_j] = 0$, provided $k + j \leq l - 1$ (in contrast to the case considered in Theorem 2).

Let ξ and η be eigenvectors of an arbitrary symplectic matrix. As is known (1), $[\xi, \eta] \neq 0$ if and only if the corresponding eigenvalues are symmetric with respect to the unit circle. In this case the vectors ξ and η are called **coskew-related**. It is clear that, for any value of the argument $t \neq t_0$ from a neighborhood of t_0 , to each eigenvector $\xi(t)$ there corresponds exactly one eigenvector $\eta(t)$ coskew-related to it.

Let $\xi(t), \eta(t)$ be a pair of mutually coskew-related eigenvectors normalized in the indefinite metric ($[\xi(t), \eta(t)] = \pm 1$).

Theorem 4. The vectors $\xi(t), \eta(t)$ belong to one and the same l_j -valued branch of a multivalued vector-function of the form (4). The function $\varphi(t) = [\xi(t), \eta(t)]$ has at the point t_0 a zero of order $(l_j - 1)/l_j$.

Suppose that all multiple eigenvalues of the matrix $X(t_0)$ lie on the unit circle. Denote by $S(t)$ the matrix whose columns are eigenvectors of the matrix $X(t)$, normalized in the indefinite metric.

Theorem 5. 1°. At the point t_0 the logarithmic derivative of the matrix $S(t)$ has a pole of first order:

$$S^{-1}dS/dt = P/(t - t_0) + C(t).$$

2°. The matrix residue P is completely determined only by the orders of the elementary divisors of the matrix $X(t_0)$ corresponding to its eigenvalues lying on the unit circle. Namely: to an elementary divisor of order l there corresponds the diagonal block

$$P_l = \frac{1}{2il} \left\| \frac{(-1)^{q-p}}{\sin \frac{\pi}{l}(q-p)} \right\|_{p,q=1}^l,$$

where in the case $q = p$ the corresponding element of the matrix P_l is replaced by zero. The elements of the residue P not included in the block P_l are zeros.

3°. The spectrum of the matrix P_l consists of the numbers:

$$\frac{l-1}{2l}, \quad \frac{l-3}{2l}, \quad \dots, \quad -\frac{l-3}{2l}, \quad -\frac{l-1}{2l},$$

and to its eigenvalue λ there corresponds the eigenvector

$$(1, \exp(-2\pi i\lambda), \exp(-4\pi i\lambda), \dots, \exp(-2(l-1)\pi i\lambda))^\tau.$$

4°. Decompose the matrix $C(t)$ into blocks in accordance with the block structure of the matrix P . A block of size $l \times m$, standing at the intersection of rows and columns included respectively in P_m and P_l , has as a singularity at the point t_0 no more than a critical pole whose order does not exceed $1 - \frac{1}{2}(1/l + 1/m)$. The remaining blocks have no critical poles.

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