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Abstract

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MATHEMATICS

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GENERALIZED EQUIPARAMETRIC MANIFOLDS IN A MULTIDIMENSIONAL PROJECTIVE SPACE

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As is known ⁽¹⁾, the problem of invariantly equipping an m -surface S_m in an n -dimensional projective space P_n is the construction of a field of $(n - m)$ -planes invariantly associated with S_m . Therefore it is of interest to study the m -parametric manifold $E(0, n - m, m)$, whose element is an $(n - m)$ -plane L_{n-m} (the basic $(n - m)$ -plane) with a fixed point L in it. In a deeper study of the geometry of S_m in P_n , an essential role is played by the manifold $E(L, \hat{L}_m, \hat{L}_{m+1})$, whose element consists of $n - m$ linearly independent $(m + 1)$ -planes $L_{\hat{m}+1}$ ($\hat{\alpha} = m + 1, \dots, n$), passing through the m -plane L_m in which the point L is given, where L_m is the tangent m -plane of the m -surface S_m described by the point L . These manifolds are the subject considered in the present work. Analogous equiparametric manifolds in three-dimensional space were studied in ⁽²⁾.

1. The derivation formulas of a certain frame, consisting of analytic points $A_0, A_1, \dots, A_n$, in the n -dimensional projective space P_n will be written in the form $dA_i = \omega_i^k A_k$ ($i, k = 0, 1, \dots, n$), where ω_i^k are Pfaffian differential forms satisfying the structure equations $D\omega_i^k = [\omega_i^j \omega_j^k]$ ($j = 0, 1, \dots, n$) and the relation $\omega_i^i = 0$. An analytic fixation of a semicanonical frame in the sense of ^(3,4) of the manifold $E(0, n - m, m)$ in P_n gives $\omega_i^k = \Lambda_{i\alpha}^k \omega_0^\alpha$ ($\alpha = 1, 2, \dots, m$), where $\Lambda_{i\alpha}^k$ satisfy the relations

$$\Lambda_{0\beta}^\alpha = \delta_\beta^\alpha, \quad \Lambda_{0\beta}^{\hat{\alpha}} = 0, \quad \Lambda_{\alpha\beta}^{\hat{\beta}} = \Lambda_{\beta\alpha}^{\hat{\beta}}, \quad A_{\underset{m-1}{0\dots 0}\hat{\alpha}} = 0, \quad A_{\underset{m-1}{\hat{\alpha}\dots\hat{\alpha}\hat{\beta}}} = 0 \quad (\hat{\alpha} \neq \hat{\beta}),$$

$$A_{\underset{m}{\hat{\alpha}\dots\hat{\alpha}}} = m, \quad \Lambda_{n\beta}^0 = 0, \quad \Lambda_{m+1,\alpha}^0 = 1$$

$$(\alpha, \beta, \gamma = 1, \dots, m; \quad \hat{\alpha}, \hat{\beta} = m + 1, \dots, n),$$

$$\Lambda_{n\beta}^\gamma \Lambda_{\gamma\alpha}^0 - \Lambda_{n\alpha}^\gamma \Lambda_{\gamma\beta}^0 = \Lambda_{\hat{\beta}\beta}^0 \Lambda_{n\alpha}^{\hat{\beta}} - \Lambda_{\hat{\beta}\alpha}^0 \Lambda_{n\beta}^{\hat{\beta}} \quad (\beta \neq \alpha, \hat{\beta} \neq n).$$

Here it is assumed that $m > 2$ and $n < m(m+1)$, and the quantities

$$m! A_{\tilde{\alpha}_1 \tilde{\alpha}_2 \dots \tilde{\alpha}_m} = \Lambda_{(\tilde{\alpha}_1|1]}^{[1} \Lambda_{\tilde{\alpha}_2|2]}^2 \dots \Lambda_{\tilde{\alpha}_m|m]}^m \quad (\tilde{\alpha}_1, \dots, \tilde{\alpha}_m = 0, m+1, \dots, n)$$

are absolutely symmetric in all indices. The manifold $E(0, n-m, m)$, referred to an arbitrary system of one-dimensional submanifolds, is determined with an arbitrariness of $m^2 + n(m-1) - 3n$ functions of m arguments. The constructed semicanonical frame has the following geometric characteristic. The point A_0 coincides with the point L , and $L_{n-m} = (A_0 A_{m+1} \dots A_n)$ is the basic $(n-m)$ -plane. The $(n-m-1)$ -plane $L_{n-m-1} = (A_{m+1} \dots A_n)$ is the $(m-1)$ -st polar, in the sense of ^(5, 6), of the point A_0 with respect to the focal algebraic hypersurface Φ_{n-m-1}^m of order m of the basic $(n-m)$ -plane. The algebraic surface Ψ_{n-m-2}^m of order m and dimension $n-m-2$, belonging simultaneously to Φ_{n-m-1}^m and

L_{n-m-1} is called a polar algebraic surface. The points $A_{\hat{\alpha}}$ ($\hat{\alpha} = m+1, \dots, n$) in L_{n-m-1} are chosen so that $A^{\hat{\alpha}} = (A_{m+1} \dots A_{\hat{\alpha}-1} A_{\hat{\alpha}+1} \dots A_n)$ is the $(m-1)$ -st polar of each of these points $A_{\hat{\alpha}}$ with respect to the polar algebraic surface. The $(m-1)$ -plane $L_{m-1} = (A_1 A_2 \dots A_m)$ belongs simultaneously to the tangent to S_m at the point A_0 , to the m -plane $L_m = (A_0 - A_m)$, and to the hyperplane Q_{n-1} passing through L_{n-m-1} and through the tangent m -plane to the m -surface described by the point A_n . The lines $A_0 A_\alpha$ are tangent to the lines described by the point A_0 under displacements along the one-dimensional coordinate submanifolds. The points A_α are the points of intersection of the lines $A_0 A_\alpha$ with the $(m-1)$ -plane L_{m-1} .

Let us note two special classes of systems of submanifolds of the manifold $E(0, n-m, m)$.

A. A system S of submanifolds, whose assignment is characterized by the relations $\Lambda_{\hat{\alpha}\beta}^\alpha = 0$, $\Lambda_{\hat{\alpha}\alpha}^\beta \neq \Lambda_{\hat{\alpha}\beta}^\alpha$ ($\alpha \neq \beta$, no summation over α and β , $\hat{\alpha}$ fixed), consists of one-dimensional submanifolds corresponding to the focal displacements of different foci of the algebraic surface Φ_{n-m-1}^m belonging to the line $A_0 A_{\hat{\alpha}}$. The canonical frame obtained by referring the manifold $E(0, n-m, m)$ to the system S is called the principal frame.

B. A system Γ of submanifolds, whose assignment is characterized by the relations $\Lambda_{\hat{\alpha}\beta}^{\hat{\alpha}} = \Lambda_{\hat{\alpha}\beta}^{\hat{\beta}} = 0$, $\Lambda_{\hat{\alpha}\alpha}^{\hat{\alpha}} \Lambda_{\hat{\beta}\beta}^{\hat{\beta}} \neq \Lambda_{\hat{\alpha}\alpha}^{\hat{\beta}} \Lambda_{\hat{\beta}\beta}^{\hat{\alpha}}$ ($\alpha \neq \beta$, $\hat{\alpha} \neq \hat{\beta}$, α and $\hat{\beta}$ fixed), consists of one-dimensional submanifolds corresponding to the focal displacements of different hyperplanes $L_{n-1}^{\hat{\alpha}, \hat{\beta}} = (L_m A_{m+1} \dots A_{\hat{\alpha}-1} A_{\hat{\alpha}+1} \dots A_{\hat{\beta}-1} A_{\hat{\beta}+1} \dots A_n, \Lambda_{\hat{\alpha}\alpha}^{\hat{\alpha}} A_{\hat{\alpha}} + \Lambda_{\hat{\beta}\beta}^{\hat{\beta}} A_{\hat{\beta}})$ (no summation over $\hat{\alpha}$ and $\hat{\beta}$), containing under these displacements the first differential neighborhood of the m -plane L_m and passing through the $(m+1)$ -planes

$L_{m+1}^{\hat{\gamma}} = (L_m A_{\hat{\gamma}})$ ($\hat{\gamma} = m+1, \dots, n$; $\hat{\gamma} \neq \hat{\alpha}, \hat{\beta}$). The canonical frame obtained by referring the manifold $E(0, n-m, m)$ to the system Γ is called a Γ -frame.

The systems S and Γ make it possible to distinguish the following special classes of manifolds $E(0, n-m, m)$.

I. A manifold $E(0, n-m, m)$ having, in the principal or Γ -frame, natural equations $\Lambda_{\alpha\alpha}^{\beta} = 0$ ($\alpha \neq \beta$), is defined with arbitrariness $(m+1)(n-m)$ functions of m arguments and is characterized by the fact that the osculating 2-planes a_{α} to the coordinate lines of the surface S_m described by the point A_0 intersect the principal $(n-m)$ -plane L_{n-m} along straight lines $l_{\alpha} = \Lambda_{\alpha\alpha}^{\hat{\alpha}}(A_0 A_{\hat{\alpha}})$, i.e., in accordance with (7), the coordinate lines of the surface S_m are geodesic lines of the projective-connection space induced by the $(n-m)$ -plane L_{n-m} along S_m .

II. A manifold $E(0, n-m, m)$ having, in the principal frame, natural equations $\Lambda_{\alpha\alpha}^{\hat{\beta}} = 0$, is defined with arbitrariness $m^2 - 2m + n$ functions of m arguments and is characterized by the fact that the coordinate lines of the surface S_m are asymptotic lines.

III. A manifold $E(0, n-m, m)$ having, in the principal frame, natural equations $\Lambda_{\alpha\alpha}^{\hat{\alpha}} = 0$, $\Lambda_{\gamma\gamma}^{\hat{\alpha}} = 0$ ($\alpha \neq \beta$, $\alpha, \beta, \gamma = 1, \dots, m$), is defined with arbitrariness $n-m$ functions of m arguments and is characterized by the fact that the coordinate lines of the surface S_m are straight lines.

2. Fixing the semicanonical frame of the manifold $E(L, L_m, L_{m+1}^{\hat{\alpha}})$, carried out analytically, gives $\omega_i^k = \lambda_{i\alpha}^k \omega^{\alpha}$, where $\Lambda_{i\alpha}^k$ satisfy the relations

$$\Lambda_{0\beta}^{\alpha} = \delta_{\beta}^{\alpha}, \quad \Lambda_{0\beta}^{\alpha} = 0, \quad \Lambda_{\alpha\beta}^{\hat{\beta}} = \Lambda_{\beta\alpha}^{\hat{\beta}}, \quad \Lambda_{\gamma\alpha}^n = 0, \quad \Lambda_{n\alpha}^{m+1} = 0, \quad A^{\hat{\alpha}\dots\hat{\alpha}} = m \quad (\hat{\gamma} \neq n),$$

$$\Lambda_{m+1,\beta}^{\alpha} \Lambda_{\alpha\gamma}^{m+1} + \Lambda_{\gamma\gamma}^{m+1} \Lambda_{n\beta}^{\hat{\gamma}} = \Lambda_{m+1,\gamma}^{\alpha} \Lambda_{\alpha\beta}^{m+1} + \Lambda_{\gamma\beta}^{m+1} \Lambda_{n\gamma}^{\hat{\gamma}} \quad (\hat{\gamma} \neq n, \gamma \neq \beta),$$

$$\Lambda_{\alpha\gamma}^n \Lambda_{\alpha\beta}^{\alpha} = \Lambda_{\alpha\gamma}^n \Lambda_{\alpha\gamma}^{\alpha}, \quad \Lambda_{\beta\alpha}^{\alpha} = 0, \quad \Lambda_{n\beta}^0 = 0, \quad \Lambda_{m+1,\alpha}^0 = 1 \quad (\beta \neq \gamma, \hat{\alpha} \neq n),$$

$$\Lambda_{n\beta}^{\gamma} \Lambda_{\gamma\tau}^0 - \Lambda_{n\tau}^{\gamma} \Lambda_{\gamma\beta}^0 = \Lambda_{\beta\beta}^0 \Lambda_{n\tau}^{\beta} - \Lambda_{\beta\tau}^0 \Lambda_{n\beta}^{\hat{\beta}} \quad (\hat{\beta} \neq m+1, \beta \neq \tau).$$

Here $m > 2$, and the quantities

$$m! A^{\hat{\alpha}_1 \dots \hat{\alpha}_m} = \Lambda_{1[1}^{\hat{\alpha}_1} \Lambda_{2]2}^{\hat{\alpha}_2} \dots \Lambda_{[m]m}^{\hat{\alpha}_m} \quad (\hat{\alpha}_{\beta} = m+1, \dots, n)$$

are absolutely symmetric in all indices. The manifold $E(0, n-m, m)$, referred to an arbitrary system of one-dimensional submanifolds, is determined with

arbitrariness $(n-m)^2 + m(m-1)$ functions of m arguments. In this frame the point A_0 coincides with the point L , $L_m = (A_0 A_1 \dots A_m)$, and $L_{m+1}^{\hat{\alpha}} = (L_m A_{\hat{\alpha}})$ ($\hat{\alpha} = m+1, \dots, n$). The lines $A_0 A_{\hat{\alpha}}$ ($\hat{\alpha} \neq n$) are B -characteristics of the $(m+1)$ -planes $L_{m+1}^{\hat{\alpha}}$ ($\hat{\alpha} \neq n$) with respect to $L_{n-1}^n = (L_m A_{m+1} \dots A_{n-1})$, i.e., the entire first differential neighborhood of each of these lines, belonging to $L_{m+1}^{\hat{\alpha}}$ ($\hat{\alpha} \neq n$), does not leave L_{n-1}^n . The line $A_0 A_n$ is a B -characteristic of the $(m+1)$ -plane L_{m+1}^n with respect to $L_{n-1}^{m+1} = (L_m A_{m+2} \dots A_n)$. Therefore the $(n-m)$ -plane $L_{n-m} = (A_0 A_{m+1} \dots A_n)$ passes through the lines $A_0 A_{\hat{\alpha}}$. Consequently, with each element of the manifold $E(L, L_m, L_{m+1}^{\hat{\alpha}})$ there is invariantly associated the manifold $E(0, n-m, m)$, and the further geometric characteristic of the frame is evident. The manifold ES_m is the manifold $E(L, L_m, L_{m+1}^{\hat{\alpha}})$ determined by the natural equations

$$A^{\hat{\alpha} \dots \hat{\alpha}} = 0 \quad (\hat{\alpha} \neq \hat{\beta}).$$

This manifold is determined with arbitrariness the product of $n-m$ functions of m arguments, where

$$m+2 < n < \frac{1}{2}m(m+3).$$

The totality of all focal hyperplanes ^(8,9) of the m -surface S_m forms a certain hypercone T_{n-1}^m of class m with vertex L_m , defined by the equation

$$T \equiv A^{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m} x_{\hat{\alpha}_1} x_{\hat{\alpha}_2} \dots x_{\hat{\alpha}_m} = 0.$$

For the manifold ES_m we have: each $(m+1)$ -plane $L_{m+1}^{\hat{\alpha}} = (L_m A_{\hat{\alpha}})$ is an $(m-1)$ -fold plane in the sense of ^(5,6) of the hyperplane

$$A^{\hat{\alpha}} = (L_m A_{m+1} \dots A_{\hat{\alpha}-1} A_{\hat{\alpha}+1} \dots A_n)$$

with respect to T_{n-1}^m . Let us note that in the manifold ES_m the surface S_m is arbitrary. Therefore the semicanonical frame of the manifold ES_m may be regarded as the semicanonical frame of an arbitrary m -surface S_m in P_n ($m+2 < n < \frac{1}{2}m(m+3)$).

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Note: Figure translations are in progress. See original paper for figures.

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