

STABILITY OF ABSTRACT BOUNDARY-VALUE PROBLEMS IN A BOUNDED DOMAIN UNDER WEAK PERTURBATIONS OF BOUNDARY OPERATORS\

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Abstract

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MATHEMATICS

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STABILITY OF ABSTRACT BOUNDARY-VALUE PROBLEMS IN A BOUNDED DOMAIN UNDER WEAK PERTURBATIONS OF BOUNDARY OPERATORS*

(Presented by Academician V. I. Smirnov on 13 V 1967)

Let

$$\mathcal{P}(x, D) = \sum_{|\alpha| \leq m} a_{\alpha}(x) D^{\alpha}$$

be a differential expression with complex-valued coefficients, given in a bounded domain Ω of Euclidean space E_n , $\Gamma = \partial\Omega^{**}$. Denote by P_0, P the minimal and maximal operators generated by $\mathcal{P}(x, D)$ in $L_2(\Omega)$, and let $\hat{P}$ ($P_0 \subset \hat{P} \subset P$; $D(\hat{P}) = \hat{D}$) be a closed extension of P_0 . Put $\hat{\alpha} = \dim \ker \hat{P}$, $\hat{\beta} = \dim \operatorname{coker} \hat{P}$.

In the present paper, under the assumption that $\hat{P}$ is normally solvable and at least one of the dimensions $\hat{\alpha}, \hat{\beta}$ is finite, sufficient conditions are obtained for the stability of the property of normal solvability and of the index of the operator $\hat{P}$ under weak, in a certain sense, perturbations of its domain of definition. The article is adjacent to the works ⁽⁵⁾, where analogous questions were treated for small perturbations.

1. Consider the Hilbert space H_P of elements $g(x) \in D(P)$ with scalar product $[g, h] = (g, h) + (Pg, Ph)$, where $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$. Analogously we construct the space $H_{\hat{P}}$ for the differential expression $\hat{\mathcal{P}}(x, D)$ formally adjoint to $\mathcal{P}(x, D)$. The normal solvability of $\hat{P}$ is equivalent to the representation ⁽⁵⁾, Lemma 1)

$$H_P = \hat{D} \dot{+} \hat{V},$$

$$\hat{V}\{v : v \in H_P, (\hat{P}\varphi, Pv) = 0, \forall \varphi \in \hat{D}; (\varphi_0, v) = 0, \forall \varphi_0 \in \ker \hat{P}\}. \quad (1)$$

Along with this, we shall consider the representation $H_{\tilde{P}} = \hat{D}^* + \hat{V}^*$ ($\hat{D}^* = D(\hat{P}^*)$, $\hat{P}^*$ is the operator adjoint to $\hat{P}$ in $L_2(\Omega)$), which is constructed from the operator $\hat{P}^*$ in the same way as the representation (1) is constructed from the operator $\hat{P}$. Denote by $\hat{\chi}$ ($\hat{\chi}^*$) the projection operator of H_P ($H_{\tilde{P}}$) onto $\hat{V}$ ($\hat{V}^*$) parallel to $\hat{D}$ ($\hat{D}^*$).

Let $\tilde{D}$ ($D(P_0) \subset \tilde{D} \subset H_P$) be another subspace and let $\tilde{P}$ be the restriction of P to $\tilde{D}$; $\tilde{D}^* = D(\tilde{P}^*)$, $\tilde{P}^*$ is the operator adjoint in $L_2(\Omega)$ to $\tilde{P}$; $\tilde{\alpha} = \dim \ker \tilde{P}$, $\tilde{\beta} = \dim \operatorname{coker} \tilde{P}$.

Theorem 1. *Let $\hat{P}$ be normally solvable. 1) If $\hat{\alpha} < \infty$ and the operator $\hat{\chi} : \tilde{D} \rightarrow L_2(\Omega)$ is completely continuous, then $\tilde{P}$ is normally solvable and $\tilde{\alpha} < \infty$. 2) If $\tilde{\beta} < \infty$ and the operator $\hat{\chi}^* : \tilde{D}^* \rightarrow L_2(\Omega)$ is completely continuous, then $\tilde{P}$ is normally solvable and $\tilde{\beta} < \infty$.*

For arbitrary $\Phi_{\pm}$ -operators in a Banach space, a similar result was obtained by G. Neubauer ((¹), Theorem 6.7). From this result the conclusions of Theorem 1 follow only under the assumption of complete continuity of the operators $\hat{\chi} : \tilde{D} \rightarrow H_P$, $\hat{\chi}^* : \tilde{D}^* \rightarrow H_{\tilde{P}}$.

* Work on this article is connected with a question posed to the author by M. Sh. Birman.

** The coefficients $a_{\alpha}(x)$ and the boundary Γ are assumed to be sufficiently smooth.

From Theorem 1 there follows the possibility of constructing a decomposition $H_P = \hat{D} + \hat{V}$ of type (1) for the operator $\hat{P}$. Let $\varkappa$ be the projection operator of H_P onto $\hat{V}$ parallel to $\hat{D}$.

Definition 1. We shall call the subspaces $\hat{D}$ and $\tilde{D}$ *completely continuous perturbations* of one another if the operators $\hat{\varkappa} : \tilde{D} \rightarrow H_P$ and $\tilde{\varkappa} : \hat{D} \rightarrow H_P$ are completely continuous.

Apparently, for the first time (and in a more general situation) this class of perturbations was studied in (¹). That, with respect to perturbations of this type, $\operatorname{ind} \hat{P}$ is not stable is shown by the following

Example. Let $N = \ker P$, $\hat{D}$ be given, $\hat{N} = \ker \hat{P}$, and $\tilde{D} = \hat{D} \oplus \{\lambda\psi\}$, where $\{\lambda\psi\}$ is a one-dimensional subspace generated by an arbitrary element $\psi \in N \ominus \hat{N}$. Then $\hat{\varkappa}\tilde{D} = \{\lambda\psi\}$, $\tilde{\varkappa}\hat{D} = \{0\}$; nevertheless $\operatorname{ind} \tilde{P} = \operatorname{ind} \hat{P} + 1$.

Denote by $\hat{V}^{\perp}$, $\tilde{V}^{\perp}$ the $[\cdot, \cdot]$ -orthogonal complements, respectively, of $\hat{D}$, $\tilde{D}$ in H_P .

Theorem 2. *Let $\hat{P}$ be a Φ ($\Phi_{\pm}$)-operator and let the subspaces $\hat{D}$, $\tilde{D}$ be completely continuous perturbations of one another. In order that $\operatorname{ind} \tilde{P} = \operatorname{ind} \hat{P}$, it is necessary and sufficient that*

$$\dim \tilde{D} \cap \hat{V}^{\perp} = \dim \hat{D} \cap \tilde{V}^{\perp}. \quad (2)$$

It is verified directly that

$$\tilde{D} \cap \hat{V}^\perp = \ker(\tilde{P}^* \hat{P} + I), \quad \tilde{D} \cap \tilde{V}^\perp = \ker(\hat{P}^* \tilde{P} + I).$$

We note that, for perturbations small in the sense of ⁽⁵⁾, both dimensions in formula (2) are equal to zero.

2. Denote by $\hat{\mathcal{K}}^\perp, \tilde{\mathcal{K}}^\perp$ the $[\cdot, \cdot]$ -orthoprojectors, respectively, onto $\hat{V}^\perp$ and $\tilde{V}^\perp$. It can be shown that the subspaces $\hat{D}, \tilde{D}$ are completely continuous perturbations of one another if and only if the operator $(\hat{\mathcal{K}}^\perp - \tilde{\mathcal{K}}^\perp) : H_P \rightarrow H_P$ is completely continuous. The latter condition is equivalent to ⁽²⁾, Lemma 3.14) the simultaneous complete continuity of the operators $R[\hat{P}] - R[\tilde{P}]$, $R[\hat{P}^*] - R[\tilde{P}^*]$, and $\hat{P}R[\hat{P}] - \tilde{P}R[\tilde{P}]$ in $L_2(\Omega)$, where, by definition,

$$R[K] = (K^*K + I)^{-1}$$

for any densely defined and closed operator K in $L_2(\Omega)$ (below K will denote an operator with these properties).

Consider in $L_2(\Omega)$ the positive self-adjoint operator

$$S[K] = (R[K])^{1/2}.$$

Lemma 1 (Cordes and Labrousse ⁽²⁾). 1) The operators K and $KS[K]$ are simultaneously normally solvable or not normally solvable. 2) $\ker K = \ker KS[K]$; $\text{coker } K = \text{coker } KS[K]$.

Definition 2. We shall call the subspaces D and $\tilde{D}$ *weak perturbations* of one another if the operator

$$(\hat{P}S[\hat{P}] - \tilde{P}S[\tilde{P}]) : L_2(\Omega) \rightarrow L_2(\Omega)$$

is completely continuous.

Theorem 3. Let $\hat{P}$ be a $\Phi(\Phi_\pm)$ -operator and let the subspaces $\hat{D}$ and $\tilde{D}$ be weak perturbations of one another. Then $\tilde{P}$ is also a $\Phi(\Phi_\pm)$ -operator and $\text{ind } \tilde{P} = \text{ind } \hat{P}$.

This theorem follows directly from Lemma 1 and the known assertions of perturbation theory for $\Phi(\Phi_\pm)$ -operators by completely continuous operators ^(3,4).

Corollary. The conclusions of Theorem 3 are valid if the operator

$$(S[\hat{P}] - S[\tilde{P}]) : L_2(\Omega) \rightarrow H_P$$

is completely continuous.

3. As in the papers ⁽⁵⁾, introduce boundary operators $\hat{A}, \hat{B}, \dots, \hat{A}'$, proceeding from the representations

$$\begin{aligned} (Pu, w) - (u, \tilde{P}w) &= \langle \hat{A}u|_\Gamma, \hat{B}w|_\Gamma \rangle + \langle \hat{B}u|_\Gamma, \hat{A}'w|_\Gamma \rangle = \\ &= \langle \tilde{A}u|_\Gamma, \tilde{B}w|_\Gamma \rangle + \langle \tilde{B}'u|_\Gamma, \tilde{A}'w|_\Gamma \rangle, \end{aligned} \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in some Hilbert space $H(\Gamma)$ of vector functions defined on Γ , such that the operators $\hat{P}, \tilde{P}$ ($\hat{P}^*, \tilde{P}^*$) are the restrictions of P ($\tilde{P}$), respectively, to the subspaces

$$\begin{aligned}\hat{D} &= \{\varphi : \hat{A}\varphi|_{\Gamma} = 0\}, \quad \tilde{D} = \{\psi : \tilde{A}\psi|_{\Gamma} = 0\} \\ (\hat{D}^* &= \{\varphi^* : \hat{A}'\varphi^*|_{\Gamma} = 0\}, \quad \tilde{D}^* = \{\psi^* : \tilde{A}'\psi^*|_{\Gamma} = 0\}).\end{aligned}$$

Put

$$\hat{Q} = (S[\hat{P}])^{-1}, \quad \tilde{Q} = (S[\tilde{P}])^{-1}, \quad T = \hat{Q} - \tilde{Q}, \quad Q = \hat{Q} + \tilde{Q}.$$

Then from the representations (3) we obtain

$$|(S[\hat{P}] - S[\tilde{P}])f|^2 = (Tf, (S[\hat{P}] - S[\tilde{P}])f) + \gamma^*; \quad (4)$$

$$\begin{aligned}\gamma &= \langle \tilde{B}'P(S[\hat{P}] - S[\tilde{P}])f|_{\Gamma}, \tilde{A}(S[\hat{P}] - S[\tilde{P}])f|_{\Gamma} \rangle \\ &+ \langle \tilde{B}'(S[\hat{P}] - S[\tilde{P}])f|_{\Gamma}, \tilde{A}'P(S[\hat{P}] - S[\tilde{P}])f|_{\Gamma} \rangle.\end{aligned} \quad (5)$$

We shall assume that the operators occurring in formulas (4), (5) have the following properties: 1) the restrictions of the operators T and Q to $C_0^\infty(\Omega)$ are pseudodifferential operators^{**}; 2) the symbols of these operators $t(x, \xi)$ and $q(x, \xi)$ admit asymptotic expansions with homogeneous in ξ terms $t_j(x, \xi)$ and $q_j(x, \xi)$ of orders, respectively, $\tau_j \searrow -\infty$ and $\chi_j \searrow -\infty$, such that the function $h(j, k) = \chi_j + \tau_k$ satisfies the condition: if $h(j_0, k_0) \geq 0$ and $j_1 + k_1 > j_0 + k_0$, then $h(j_1, k_1) < h(j_0, k_0)$; 3) if the sequences $\{\varphi_n\} \subset \hat{D}$ and $\{\psi_n\} \subset \tilde{D}$ are such that $\{\varphi_n - \psi_n\}$ is bounded in H_P , then $\{\tilde{B}'(\varphi_n - \psi_n)|_{\Gamma}\}$ is a bounded sequence and $\{\hat{A}\varphi_n|_{\Gamma}\}$ is a compact sequence in $H(\Gamma)$; 4) if the sequences $\{\varphi_n^*\} \subset \hat{D}^*$ and $\{\psi_n^*\} \subset \tilde{D}^*$ are such that $\{\varphi_n^* - \psi_n^*\}$ is bounded in H_P , then $\{\tilde{B}'(\varphi_n^* - \psi_n^*)|_{\Gamma}\}$ is a bounded sequence and $\{\hat{A}'\varphi_n^*|_{\Gamma}\}$ is a compact sequence in $H(\Gamma)$.

Lemma 2. *Let conditions 1), 2) be fulfilled. Then the operator*

$$T : C_0^\infty(\Omega) \cap L_2(\Omega) \rightarrow L_2(\Omega)$$

is completely continuous.

Remark. If $j_1 + k_1 > j_0 + k_0$ implies $h(j_1, k_1) < h(j_0, k_0)$ for all integers j_0, k_0 , then $T \in \mathcal{L}_{-\infty}$ (i.e. $t_j(x, \xi) \equiv 0$, $j = 0, 1, 2, \dots$).

Lemma 2 and the remark to it are derived from the obvious relation $QT = -TQ$ by constructing an asymptotic series for the symbol of the operator QT .

Theorem 4. *Let the conditions 1)–4) formulated above be fulfilled. Then the subspaces $\hat{D}$ and $\tilde{D}$ are weak perturbations of one another, the operators $\hat{P}$ and $\tilde{P}$ either both are not, or both are, Φ -($\Phi_{\pm}$)-operators, and in the latter case*

$$\text{ind } \tilde{P} = \text{ind } \hat{P}.$$

The proof of this theorem is based on the corollary to Theorem 3, Lemma 2, and an analysis of formulas (4), (5).

Let us note that the conclusions of Lemma 2 and Theorem 4 remain valid if conditions 1), 2) are replaced by the following: on $C_0^\infty(\Omega)$ the operators $\hat{Q}, \tilde{Q}$ and Q are elliptic pseudodifferential operators of order m , while $S[\hat{P}], S[\tilde{P}]$, and Q^{-1} are of order $-m$.

In the case when the coefficients $a_\alpha(x)$ are constant, condition 1) is fulfilled, and $T \in \mathcal{L}_{-\infty}$. Thus, in this case the conclusions of Theorem 4 follow from only the conditions 3), 4) on the boundary operators.

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* $|\cdot|$ is the norm of H_P .

** In the sense of (6, 7).

Note: Figure translations are in progress. See original paper for figures.

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