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Abstract

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MATHEMATICS

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AXIOMATICS OF MEASURE THEORY IN LINEAR SPACES

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1°. Usually, in studying measure theory in linear spaces one starts from the assumption that the space has one or another locally convex topology. However, a careful analysis shows that a considerable part of the problems of this theory, both internal and arising from probability theory and other applications, is not connected with the topology of the space and is of a purely metric nature. Such problems include questions on the absolute continuity of measures, the theory of transformations with invariant and quasi-invariant measure in a linear space, and others. It turned out that the “metric” point of view makes it possible to see more clearly the essence both of these problems and of problems connected with topology (for example, questions on extension of a measure). In order to carry out the “metric” program fully, it is desirable to describe axiomatically the structure of a linear space with a measure, without resorting to topology in order to establish connections between the structure of the linear space and that of the measure space. This is what is done in the present note.

The decisive role in the proposed theory is played by the concept of a measurable linear functional and by the duality between a linear space with measure in the sense of the axiomatics given here and the space of all linear measurable functionals (see ⁽¹⁾). This duality makes measure theory in linear spaces a peculiar chapter of the classical theory of functions of a real variable and of general measure theory.

The main result of the note consists in proving the existence of a sufficient number of linear measurable functionals in linear spaces with measure and the resulting duality theorem. The proof is based on an analysis of the very important concept in its own right of a linear space with free measure, introduced in ⁽²⁾ and used there for other purposes.

In ⁽¹⁾ duality is applied to obtain the basic facts on quasi-invariant measures and for other questions. In subsequent publications it is intended to apply it to the problem of extension of measure, to the theory of transformations of measures in linear spaces, and others.

2°. Axiomatics. A **linear space with measure** (l.s.m.) is a pair (E, μ) satisfying three axioms:

I. E is a linear space over the field of real numbers.

II. (E, μ) is a space with a normalized nonnegative countably additive measure, defined on a complete sigma-algebra, and is (as a measure space) a Lebesgue–Rokhlin space ⁽³⁾.

Denote by $\xi_E^{\lambda_1, \dots, \lambda_n}$ (λ_i are real numbers, $i = 1, \dots, n$) the partition of the direct product $\underbrace{E \times \dots \times E}_n$ into inverse images of points under the mapping

$$\Gamma_n^{\lambda_1, \dots, \lambda_n} : \underbrace{E \times \dots \times E}_n \rightarrow E^n, \quad \Gamma_n^{\lambda_1, \dots, \lambda_n}(x_1, \dots, x_n) = \sum_{i=1}^n \lambda_i x_i$$

and introduce on $\underbrace{E \times \dots \times E}_n$ the measure $\underbrace{\mu \times \dots \times \mu}_n$, which is the usual direct product of measures; $(\underbrace{E \times \dots \times E}_n, \underbrace{\mu \times \dots \times \mu}_n)$, as a space with measure, is again a Lebesgue–Rokhlin space.

III. $\xi_E^{\lambda_1, \dots, \lambda_n}$ is a measurable partition of $(\underbrace{E \times \dots \times E}_n, \underbrace{\mu \times \dots \times \mu}_n)$ in the sense of (3) for any set $\{\lambda_i\}_{i=1}^n$ of real numbers, $n = 1, 2, \dots$

The geometric meaning of axiom III consists in the possibility of defining the notion of a **generalized convolution**—this is what we shall call the measure $\mu^{\lambda_1, \dots, \lambda_n}$ on E , which is the image of the measure $\underbrace{\mu \times \dots \times \mu}_n$ on $\underbrace{E \times \dots \times E}_n$ under the mapping

$$\Gamma_n^{\lambda_1, \dots, \lambda_n} : \underbrace{E \times \dots \times E}_n \rightarrow E$$

(if $\lambda_i = 1$, $i = 1, 2, \dots, n$, then $\mu^{\overbrace{1, \dots, 1}^n}$ is the ordinary convolution). The pair $(E, \mu^{\lambda_1, \dots, \lambda_n})$, as is easy to show, is again an l.m.s. in our sense.

The most widespread example of an l.m.s. is as follows. Let E be a complete metrizable locally convex real space in which there exists a countable total set of continuous linear functionals; μ a measure given on the σ -algebra of weak Borel sets; axiom III is then satisfied automatically. Another example of an l.m.s.: E is a complete separable linear metric space (not necessarily locally convex), μ is a Borel measure ^{*}.

Let us note that no more than a countable product of l.m.s. is an l.m.s.

A **linear homomorphism** (l.h.) T of an l.m.s. (E_1, μ_1) into an l.m.s. (E_2, μ_2) is a homomorphism T in the sense of measure theory such that

$$T\mu_1 = \mu_2 \quad \text{and} \quad T\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i T x_i$$

for almost all, with respect to the measure $\underbrace{\mu \times \dots \times \mu}_n$, sets $\{x_i\}_{i=1}^n$, and for any (but fixed) real numbers $\{\lambda_i\}_{i=1}^n$, $n = 1, \dots$. If T is an l.h., then $\xi_{E_1}^{\lambda_1, \dots, \lambda_n}$ is a refinement of the partition $(T \times \dots \times T)^{-1} \xi_{E_2}^{\lambda_1, \dots, \lambda_n}$. An l.h. T is called a **linear isomorphism** (l.i.) if T is an isomorphism in the sense of measure theory and, for any $\{\lambda_i\}_{i=1}^n$, $n = 1, 2, \dots$,

$$(T \times \dots \times T)^{-1} \xi_{E_1}^{\lambda_1, \dots, \lambda_n} = \xi_{E_2}^{\lambda_1, \dots, \lambda_n} \mod 0.$$

Let us note that an l.h. may be an isomorphism in the sense of measure theory and not be an l.i. It follows from Theorem 1 (Corollary 1) that every l.h. mod 0 is linear in the ordinary sense (i.e. additive and homogeneous) on a linear set of full measure, which justifies our terminology.

Two l.m.s. are called **linearly isomorphic** (linearly isomorphic mod 0) if there exists a linear isomorphism (linear isomorphism mod 0) connecting them. A **linear measurable functional** (l.m.f.) on an l.m.s. is an l.h. that is a measurable numerical function on the l.m.s. We shall say that on an l.m.s. (E, μ) there exists a **sufficient number** of l.m.f.'s if the minimal σ -algebra with respect to which all l.m.f.'s are measurable coincides mod 0 with the whole σ -algebra on which the measure is given. Let us give an equivalent formulation. For this we introduce the linear topological ring, basic for all functional considerations, $S_\mu(E)$ —the classes of all real measurable functions on (E, μ) that coincide mod 0, with the (metrizable) topology of convergence in measure and ordinary multiplication. A set in $S_\mu(E)$ is called **generating** if the minimal closed subring containing this set coincides with the whole ring $S_\mu(E)$. We now denote the set

* Direct establishment of Theorem 1 already in this case is quite difficult.

For one special case this was done by V. N. Sudakov.

(classes) of l.f.'s on (E, μ) by $\mathcal{L}_\mu$; it is nonempty, since $0 \in \mathcal{L}_\mu$, and forms a closed linear subspace in $S_\mu(E)$. The required formulation is as follows: $\mathcal{L}_\mu$ is generating in $S_\mu(E)$.

A **measurable affine partition** of an l.p.m. (E, μ) with respect to a linear manifold $K \subset E$ is called a partition $E_\eta = \eta_{E,K}$ into translates of K , if: 1) $\eta_{E,K}$ is measurable in (E, μ) , and 2) for any set of real numbers $\{\lambda_i\}_{i=1}^n$, $n = 1, 2, \dots$, the partition $\eta_{E,K}$ is measurable in $(E, \mu^{\lambda_1, \dots, \lambda_n})$. The quotient space $(E/\eta_{E,K}, \mu/\eta_{E,K})$ is canonically an l.p.m., and the canonical homomorphism is an l.g.

3°. Free measure and the existence of a sufficient number of l.i.f. An l.p.m. $(\mathcal{E}, \rho)$ is called a **linear space with free measure** (l.s.f.m.) if every measurable real-valued function on $(\mathcal{E}, \rho)$ coincides mod 0 with an l.i.f. One

can give a direct definition of an l.s.f.m. in terms of partitions $\eta_{\mathcal{E}}^{\lambda_1, \dots, \lambda_n}$. If ρ is a discrete measure, then the corresponding l.s.f.m. is linearly isomorphic to (R^N, ρ_1) , where ρ and ρ_1 have the same metric type ⁽³⁾ and ρ_1 is concentrated at the ends of the unit basis vectors, while N is the number of atoms of the measure ρ . If ρ is a continuous measure, then the actual construction of an l.s.f.m. is rather laborious; it is carried out in ⁽²⁾.

We shall need the following properties of l.s.f.m.

A. For a given metric type of measure there exists, up to linear isomorphism, exactly one l.s.f.m. This isomorphism can be chosen so that it is additive and homogeneous on a linear set of full measure. An l.s.f.m. with continuous measure can be realized as R^∞ (or as a separable Hilbert space, see ⁽²⁾) with a certain Borel measure.

B. Every l.i.f. on an l.s.f.m. is mod 0 additive and homogeneous on a linear set of full measure.

C. If $\eta_{\mathcal{E}, K_1}^{\mathcal{E}} = \eta_1$ and $\eta_{\mathcal{E}, K_2}^{\mathcal{E}} = \eta_2$ are two measurable affine partitions of an l.s.f.m. $(\mathcal{E}, \rho)$, and $\eta_1 = \eta_2 \pmod{0}$ with respect to $\rho^{\lambda_1, \dots, \lambda_n}$ for all $\{\lambda_i\}_{i=1}^n$, $n = 1, 2, \dots$, then the l.p.m. $(\mathcal{E}/\eta_1, \rho/\eta_1)$ and the l.p.m. $(\mathcal{E}/\eta_2, \rho/\eta_2)$ are linearly isomorphic, and the isomorphism is additive and homogeneous on a linear set of full measure.

Theorem 1. In every l.p.m. (E, μ) there exists a sufficient number of l.i.f.

Proof is based on the following idea. Form a linear space $\mathcal{E}$ which is the free real vector space over E ($\mathcal{E}$ is the set of formal linear combinations of elements of E). Let $\tau : E \rightarrow \mathcal{E}$ be the natural embedding. Transfer to $\mathcal{E}$ the metric structure from (E, μ) by means of τ . It is easy to verify that $(\mathcal{E}, \rho)$ ($\rho = \tau\mu$) is an l.s.f.m. Let $\beta : \mathcal{E} \rightarrow E$ be the linear extension of τ^{-1} , and $K = \beta^{-1}(0)$. Then η_K is an affine measurable partition of $(\mathcal{E}, \rho)$, and $(\mathcal{E}/\eta_K, \rho/\eta_K)$ is linearly isomorphic to (E, μ) . Combining properties A-C, we obtain the existence of a sufficient number of l.i.f. on (E, μ) .

Corollary 1. Every l.g. $T : (E_1, \mu_1) \rightarrow (E_2, \mu_2)$ (in particular, an l.i.f.) coincides mod 0 with an additive homogeneous isomorphism defined on a linear set of full measure. This set, generally speaking, depends on the l.g.

Corollary 2. Every l.p.m. (E, μ) is linearly isomorphic to an l.p.m. (R^∞, ν_1) or to an l.p.m. $(\mathcal{H}, \nu_1)$, where $\mathcal{H}$ is a separable Hilbert space, and ν_1 and ν_2 are Borel measures respectively in R^∞ and $\mathcal{H}$.

Corollary 2 shows that the topology of an l.p.m. is not essential in the metric theory of l.p.m. Theorem 1 makes it possible to give the following equivalent definition of an l.p.m.: a pair (E, μ) is called an l.p.m. if it satisfies axioms I, II of item 2° and the axiom:

III'. There exists a sufficient number of l.i.f. on (E, μ) . Here, by an l.i.f. one may (Corollary 1) understand a measurable function on (E, μ) , additive and homogeneous on a linear set of full measure.

4°. Duality. In the ring $S_\mu(E')$ there is given a characteristic functional

$$\chi(f) = \int_E e^{if(x)} d\mu, \quad f \in S_\mu(E').$$

Theorem 2. Let T be a l.g. l.p.m. (E_1, μ_1) into a l.p.m. (E_2, μ_2) . Then the homomorphism of rings

$$U_T : S_{\mu_2}(E'_2) \rightarrow S_{\mu_1}(E'_1),$$

acting by the formula

$$(U_T f)(x) = f(Tx)$$

takes $\mathcal{L}_{\mu_2}$ into $\mathcal{L}_{\mu_1}$; if T is l.i., then $U_T \mathcal{L}_{\mu_2} = \mathcal{L}_{\mu_1}$. Conversely, if

$$U : S_{\mu_2}(E'_2) \rightarrow S_{\mu_1}(E'_1)$$

is a continuous multiplicative linear operator preserving the characteristic functional and such that

$$U \mathcal{L}_{\mu_2} \subset \mathcal{L}_{\mu_1} \quad (U \mathcal{L}_{\mu_2} = \mathcal{L}_{\mu_1}),$$

then there exists a l.e. (l.i.) $T : (E_1, \mu_1) \rightarrow (E_2, \mu_2)$ such that $U = U_T$; T can be chosen so that it is linearly additive and homogeneous on a set of full measure.

The proof is based on Corollary 3. The duality mentioned in item 1 is most easily formulated in the terms of category theory.

The objects of the category Linmes are l.p.m.; a morphism is an l.g. mod 0. The objects of the category SL are linear topological rings of all classes, coinciding mod 0, of real measurable functions on a Lebesgue-Rokhlin space (with the topology of convergence in measure and ordinary multiplication), with a distinguished closed generating subspace; morphisms are homomorphisms of rings (linear continuous multiplicative operators defined everywhere) preserving the characteristic functional and taking the distinguished subspace into the distinguished one*.

By Theorem 1, if (E, μ) is an object of Linmes, then $(S_\mu(E), \mathcal{L}_\mu)$ is an object of SL. By Theorem 2, the correspondence

$$d : (E, \mu) \sim (S_\mu(E), \mathcal{L}_\mu)$$

is a contravariant functor from the category Linmes to the category SL.

Theorem 3. The category dual to Linmes, $(\text{Linmes})^0$, is equivalent to the category SL (for definitions see (4)). The equivalence is established by means of the functor d^0 , considered on $(\text{Linmes})^0$.

It follows from Theorem 3 that every fact concerning l.p.m. can be expressed in dual terms of the space of linear measurable functionals. The latter plays

for the theory of l.p.m. the role of the conjugate space in the theory of linear topological spaces. Theorem 3 shows that in the most interesting case—the case of continuous measure—the theory of l.p.m. is the theory of closed generating subspaces of the ring $S_m([0, 1])$, where m is Lebesgue measure on $[0, 1]**$. For their study the classical methods are useful—the theory of orthogonal series, characteristic functions, etc. These considerations have long been known in probability theory; however, the specific feature of the problems considered here consists in the fact that no basis is fixed in the subspace L : we are interested only in the invariant properties of L .

In conclusion we note that slight modifications of the arguments presented make it possible to give an axiomatization of commutative groups (groups with operators, modules) with measure and to prove for them the existence of a sufficient number of measurable characters. It would be interesting to investigate non-commutative groups with measure.

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* It is useful to note that morphisms of the category SL in fact preserve not only the ring and topological structure, but also the partial order present in the ring S ; moreover, they take L^p into L^p and are isometric there. These properties may be adopted as their definition.

** Recall that a Lebesgue-Rokhlin space with continuous measure is isomorphic, in the sense of measure theory, to the space $([0, 1], m)$.

Note: Figure translations are in progress. See original paper for figures.

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