

# ON A LINEAR FORM IN LOGARITHMS OF ALGEBRAIC NUMBERS

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## Abstract

## Full Text

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MATHEMATICS

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# ON A LINEAR FORM IN LOGARITHMS OF ALGEBRAIC NUMBERS

(Presented by Academician L. I. Sedov on 29 II 1968)

By the **length**  $L(\xi)$  of an algebraic number  $\xi$  we shall mean the sum of the moduli of the coefficients of the equation defining the number  $\xi$ .

Let  $K$  be the field obtained by adjoining the algebraic numbers  $\alpha_1, \dots, \alpha_m; \beta_0, \beta_1, \dots, \beta_m$  to the field of rational numbers, and let  $n$  be its degree,

$$h = \max\{L(\alpha_1), \dots, L(\alpha_m), e^{|\ln \alpha_1|}, \dots, e^{|\ln \alpha_m|}\},$$

$$H = \max L(\beta_i), \quad i = 0, 1, \dots, m.$$

There exists an effective absolute constant  $c_0$  such that, if the quantities  $\ln \alpha_1, \dots, \ln \alpha_m$  are linearly independent over the field of rational numbers, and if  $|\beta_0| + |\beta_1| + \dots + |\beta_m| > 0$ , then the inequality

$$|\beta_0 + \beta_1 \ln \alpha_1 + \dots + \beta_m \ln \alpha_m| > \exp \left[ -(c_0 + 90^{m^2} n \ln h)^{16m^2 \ln H} \right]. \quad (1)$$

holds.

Interest in effective estimates of type (1) arose long ago, since, besides their independent significance, they make it possible to effectivize the solution of a number of problems in number theory (see, for example, <sup>(1)</sup>).

For  $m = 2$ , for a homogeneous form a similar estimate was first obtained in 1935 by A. O. Gelfond. For its derivation he used his interpolation method, created by him for the solution of the well-known seventh Hilbert problem. Successively strengthening his estimate, A. O. Gelfond obtained <sup>(2)</sup> the inequality

$$|\beta_1 \ln \alpha_1 + \beta_2 \ln \alpha_2| > \exp[-n^2 \ln^{2+\varepsilon}(n + \ln H)], \quad (2)$$

$$\varepsilon > 0, \quad n + \ln H \geq H_0(\ln \alpha_1, \ln \alpha_2, \varepsilon).$$

For an arbitrary natural  $m$ , A. O. Gelfond obtained <sup>(3)</sup> the inequality

$$|x_1 \ln \alpha_1 + \dots + x_m \ln \alpha_m| > e^{-\varepsilon x},$$

$$\varepsilon > 0, \quad x = \max_{1 \leq i \leq m} |x_i| \geq x_0(\ln \alpha_1, \dots, \ln \alpha_m, \varepsilon).$$

Here  $x_1, \dots, x_m$  are rational integers. This inequality was obtained by the Thue-Siegel method, so that the constant  $x_0$  is ineffective.

The first effective estimates of the form  $\beta_1 \ln \alpha_1 + \dots + \beta_m \ln \alpha_m$  were obtained by A. Baker <sup>(4-6)</sup>. In <sup>(4)</sup> he obtained the inequality

$$|\beta_1 \ln \alpha_1 + \dots + \beta_m \ln \alpha_m| > C \exp[-\ln^\chi H], \quad (3)$$

where  $\chi > m + 1$ , the constant  $C$  depends on  $\chi, n, \ln \alpha_1, \dots, \ln \alpha_m$ , and the numbers  $\ln \alpha_1, \dots, \ln \alpha_m$  and  $\pi i$  are linearly independent over the field of rational numbers.

In <sup>(5)</sup> linear independence is required only for the numbers  $\ln \alpha_1, \dots, \ln \alpha_m$  themselves, but here  $\chi > 2m + 1$ . In <sup>(6)</sup> A. Baker extended his results to nonhomogeneous forms.

A. Baker proved his theorems by means of the above-mentioned method of A. O. Gelfond, adding to it an important additional consideration concerning the possibility of "many-fold" interpolation of the auxiliary form.

For the derivation of inequality (1), use is made primarily of A. O. Gelfond's method, as well as of A. Baker's additions and some further considerations.

In deriving inequality (1), the author sought to obtain the best estimate with respect to  $H$ . This "preference" for  $H$  corresponds to the case of fixed  $\ln \alpha_1, \dots, \ln \alpha_m$  and bounded  $n$  (it is precisely this case that was considered in the works of A. O. Gelfond and A. Baker), or to the case in which  $h$  and  $n$  vary sufficiently slowly relative to  $H$ . In the case where all the quantities  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$  vary independently, one can improve the estimate with respect to  $h$  and  $n$  by worsening it with respect to  $H$ .

An inequality analogous to inequality (1) is also valid for the  $p$ -adic metric.

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*Note: Figure translations are in progress. See original paper for figures.*

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