

ON THE ACCURACY OF THE APPROXIMATION OF THE DISTRIBUTION OF A SUM OF INDEPENDENT RANDOM VARIABLES TO THE NORMAL DISTRIBUTION

L. V. Osipov

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.78220>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.21

MATHEMATICS

L. V. OSIPOV

ON THE ACCURACY OF THE APPROXIMATION OF THE DISTRIBUTION OF A SUM OF INDEPENDENT RANDOM VARIABLES TO THE NORMAL DISTRIBUTION

(Presented by Academician Yu. V. Linnik on 13 IV 1967)

1. Consider a sequence of independent identically distributed random variables $X_1, \dots, X_n, \dots$ with common distribution function $V(x)$, characteristic function $v(t)$, and positive variance $\sigma^2 = DX_n < \infty$. Without loss of generality one may assume that $\mathbf{E}X_n = 0$. Denote by $F_n(x)$ the distribution function of the normalized sum

$$\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^n X_j,$$

and by $\Phi(x)$ the normal distribution function with zero mean and unit variance.

It is known that $\sup_x |F_n(x) - \Phi(x)| \rightarrow 0$ as $n \rightarrow \infty$. The asymptotic behavior of $F_n(x) - \Phi(x)$ as $n \rightarrow \infty$ has been studied by many authors. Under the assumption of the existence of the third moment $a_3 = \mathbf{E}X_n^3$, it follows from the known asymptotic expansions of the function $F_n(x)$ ⁽¹⁾ that

$$\sup_x |F_n(x) - \Phi(x)| = n^{-1/2}(A + o(1)) \quad (n \rightarrow \infty), \quad (1)$$

where $A = |a_3|/6\sqrt{2\pi}\sigma^3$ if X_n has a nonlattice distribution, and $A = |a_3|/6\sqrt{2\pi}\sigma^3 + h/2\sqrt{2\pi}\sigma$ if X_n has a lattice distribution with maximal span h .

We shall consider the case where $\mathbf{E}|X_n|^3 = \infty$. If, in addition, the condition $\limsup_{|t| \rightarrow \infty} |v(t)| < 1$ (Cramér's condition (C)) is satisfied, then we obtain the relation $\sup_x |F_n(x) - \Phi(x)| \asymp \psi_{n,2}$, where the quantity $\psi_{n,2}$ is a certain functional of the function $nV(x\sigma\sqrt{n})$ (Theorem 1)*. In the case of lattice distributions the relation

$$\sup_x |F_n(x) - \Phi(x)| \asymp \left(\psi_{n,2} + \frac{1}{\sqrt{n}} \right)$$

holds (Theorem 2). In Theorems 3 and 4 analogous results are obtained for the remainder term in the asymptotic expansion of the function $F_n(x)$.

The methods of the present paper are those of papers ^(1,2).

2. We formulate the main results. Put

$$\psi_{n,2} = \frac{1}{\sigma^2} \int_{|x| > \sigma\sqrt{n}} x^2 dV(x) + \frac{1}{\sigma^3\sqrt{n}} \left| \int_{|x| \leq \sigma\sqrt{n}} x^3 dV(x) \right| + \frac{1}{\sigma^4 n} \int_{|x| \leq \sigma\sqrt{n}} x^4 dV(x).$$

Theorem 1. If $\mathbf{E}|X_n|^3 = \infty$ and $\limsup_{|t| \rightarrow \infty} |v(t)| < 1$, then

$$\sup_x |F_n(x) - \Phi(x)| \asymp \psi_{n,2}. \quad (2)$$

* The relation $a_n \asymp b_n$ means that the sequences a_n and b_n satisfy the relation $0 < \liminf_{n \rightarrow \infty} a_n/b_n \leq \limsup_{n \rightarrow \infty} a_n/b_n < \infty$.

Theorem 2. If X_n has a lattice distribution, then

$$\sup_x |F_n(x) - \Phi(x)| \asymp (\psi_{n,2} + 1/\sqrt{n}). \quad (3)$$

Let us note that the assertion of Theorem 2 in the case when $\mathbf{E}|X_n|^3 < \infty$ follows from the stronger relation (1).

Suppose now that $\mathbf{E}|X_n|^3 < \infty$ for some integer $k \geq 3$. Introduce the notation:

$$\Lambda_{n,\nu} = \frac{1}{\sigma^\nu n^{(\nu-2)/2}} \int_{|x| > \sigma\sqrt{n}} x^\nu dV(x) \quad (\nu = 1, \dots, k),$$

$$L_{n,\nu} = \frac{1}{\sigma^\nu n^{(\nu-2)/2}} \int_{|x| < \sigma\sqrt{n}} x^\nu dV(x) \quad (\nu = 1, 2, \dots),$$

$$\psi_{n,k} = \begin{cases} \Lambda_{n,k} + |L_{n,k+1}| + L_{n,k+2}, & \text{if } k \text{ is even,} \\ \Lambda_{n,k-1} + |L_{n,k}| + L_{n,k+1}, & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 3. Let $\limsup_{|t| \rightarrow \infty} |v(t)| < 1$, and let there exist an integer $k \geq 3$ such that $\mathbf{E}|X_n|^k < \infty$, $\mathbf{E}|X_n|^{k+1} = \infty$. Then, for odd k ,

$$\sup_x \left| F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{P_\nu(-\Phi)}{n^{\nu/2}} \right| \asymp \psi_{n,k},$$

and, for even k ,

$$\sup_x \left| F_n(x) - \Phi(x) - \sum_{\nu=1}^{k-2} \frac{P_\nu(-\Phi)}{n^{\nu/2}} - \frac{P'_{k-1}(-\Phi)}{n^{(k-1)/2}} \right| \asymp \psi_{n,k}.$$

Here $P_\nu(-\Phi)$ are the functions known in the theory of probabilistic asymptotic expansions (see ⁽³⁾). Namely,

$$P_\nu(-\Phi) = \sum \prod_{m=1}^{\nu} \frac{1}{r_m!} \left(\frac{\gamma_{m+2}}{\sigma^{m+2}(m+2)!} \right)^{r_m} W_{3r_1+\dots+(\nu+2)r_\nu}(x), \quad (4)$$

where the summation is over all nonnegative integer solutions of the equation $r_1 + 2r_2 + \dots + \nu r_\nu = \nu$, γ_ν is the cumulant of order ν of the random variable X_n ,

$$W_s(x) = -\frac{s!}{\sqrt{2\pi}} e^{-x^2/2} \sum_{m=0}^{[s/2]} \frac{(-1)^m x^{s-2m}}{m!(s-2m)!2^m}.$$

The cumulants of the random variable X_n are expressed in the following way through its moments $\alpha_m = \mathbf{E}X_n^m$:

$$\gamma_\nu = \nu! \sum (-1)^{r_1+\dots+r_\nu-1} (r_1 + \dots + r_\nu - 1)! \prod_{m=1}^{\nu} \frac{1}{r_m!} \left(\frac{\alpha_m}{m!} \right)^{r_m}, \quad (5)$$

where the summation is over all nonnegative integers $r_1, r_2, \dots, r_\nu$ satisfying the equation $r_1 + 2r_2 + \dots + \nu r_\nu = \nu$. Under the conditions of Theorem 3, γ_ν are defined only for $\nu \leq k$, and the functions $P_\nu(-\Phi)$ only for $\nu \leq k-2$. For large values of ν we shall regard (4) and (5) as formal equalities. We set the function $P'_{k-1}(-\Phi)$ equal to the sum (4) for $\nu = k-1$, in which, from γ_{k+1} , we omit the term containing α_{k+1} .

Let us formulate an analogous result for lattice distributions. Introduce the functions $S_\nu(x)$, setting

$$\begin{aligned} S_1(x) &= \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{\pi m}, & S_2(x) &= \sum_{m=1}^{\infty} \frac{\cos 2\pi m x}{2(\pi m)^2}, \dots \\ \dots, & S_{2l}(x) = \sum_{m=1}^{\infty} \frac{\cos 2\pi m x}{2^{2l-1}(\pi m)^{2l}}, & S_{2l+1}(x) &= \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{2^{2l}(\pi m)^{2l+1}}, \dots \end{aligned}$$

Next, let

$$\Pi_{n,2}(x) = \Phi(x), \quad \Pi_{n,l}(x) = \Phi(x) + \sum_{\nu=1}^{l-2} \frac{P_{\nu}(-\Phi)}{n^{\nu/2}} \quad (l = 3, \dots, k),$$

$$\delta_{\nu} = \begin{cases} 1, & \text{if } \nu = 4m + 1, 4m + 2, \\ -1, & \text{if } \nu = 4m + 3, 4m. \end{cases}$$

Theorem 4. Suppose X_n assumes, with positive probabilities, only values of the form $a + sh$ ($s = 0, \pm 1, \dots$), where a is some real number and h is the maximal span of the distribution. If $\mathbf{E}|X_n|^k < \infty$ for some integer $k \geq 3$, then

$$\sup_x |F_n(x) - \Pi_{n,k}(x) - \sum_{\nu=1}^{k-2} \delta_{\nu} \left(\frac{h}{\sigma\sqrt{n}} \right)^{\nu} S_{\nu} \left(\frac{x\sigma\sqrt{n}}{h} - \frac{na}{h} + \left[\frac{na}{h} \right] \right) \frac{d^{\nu}}{dx^{\nu}} \Pi_{n,k-\nu}(x)| > (\psi_{n,k} + n^{-(k-1)/2}).$$

In the case where $\mathbf{E}|X_n|^{k+1} < \infty$, Theorem 4 follows from the known estimate of the remainder term in the asymptotic expansion of $F_n(x)$ ⁽¹⁾.

3. We outline the proof of Theorems 1 and 2. By $C_1, C_2, \dots$ we shall denote certain positive constants not depending on n . Put

$$\Delta_n = \sup_x |F_n(x) - \Phi(x)|.$$

The proof of the upper estimate for Δ_n in Theorem 1 is based on applying the theorem of C. G. Esseen ⁽¹⁾, p. 32) with $F(x) = F_n(x)$, $G(x) = \Phi(x)$, $T = n$, and on the following expansion of the logarithm of the function $f_n(t) = v^n(t/\sigma\sqrt{n})$:

$$\ln f_n(t) = -t^2/2 + (t^2 + t^4)O(\psi_{n,2} + 1/n) \quad (n \rightarrow \infty)$$

uniformly in t in the interval $|t| < \sqrt{n}$. The proof of the upper estimate for Δ_n in Theorem 2 is carried out analogously; in this case we put $T = C_1\sqrt{n}$.

In proving the lower estimates for Δ_n , the methods of the work of I. A. Ibragimov ⁽²⁾ are used. Consider bounded functions $A(x)$ and $B(t)$ such that

$$\int |A(x)| dx < \infty, \quad B(t) = \int e^{itx} A(x) dx.$$

It is not difficult to see that the functions $F_n(x) - \Phi(x)$ and $(f_n(t) - e^{-t^2/2})/(-it)$ belong to $L_2(-\infty, \infty)$ and constitute a Fourier-transform pair. According to Parseval's identity, we have

$$-2\pi \int (F_n(x) - \Phi(x)) \overline{A(x)} dx = \int (f_n(t) - e^{-t^2/2}) \frac{\overline{B(t)}}{it} dt.$$

Consequently,

$$\Delta_n \asymp C_2 \left| \int (f_n(t) - e^{-t^2/2}) \frac{\overline{B(t)}}{it} dt \right|. \quad (6)$$

Further, uniformly with respect to t in the interval $|t| < T_n = \min(\psi_{n,2}^{-1/4}; n^{1/4})$,

$$f_n(t) - e^{-t^2/2} = e^{-t^2/2} \left[n \left(v \left(\frac{t}{\sigma\sqrt{n}} \right) - 1 + \frac{t^2}{2n} \right) + (t^4 + t^8) O \left(\psi_{n,2}^2 + \frac{1}{n} \right) \right].$$

Let $|B(t)| < C_3 e^{-t^2/4}$. Then the integral on the right-hand side of (6) is equal to

$$\begin{aligned} & n \int \frac{\overline{B(t)}}{it} e^{-t^2/2} \left(v \left(\frac{t}{\sigma\sqrt{n}} \right) - 1 + \frac{t^2}{2n} \right) dt + O \left(\psi_{n,2}^2 + \frac{1}{n} \right) = \\ & = n \iint \frac{\overline{B(t)}}{it} e^{-t^2/2} \left(e^{itx/\sigma\sqrt{n}} - 1 + \frac{t^2 x^2}{2\sigma^2 n} \right) dt dV(x) + O \left(\psi_{n,2}^2 + \frac{1}{n} \right) = \\ & = I + O \left(\psi_{n,2}^2 + \frac{1}{n} \right). \end{aligned}$$

Putting here $B(t) = -ite^{-t^2/2}$, we find

$$I = n\sqrt{\pi} \int \left(e^{-x^2/4\sigma^2 n} - 1 + \frac{x^2}{4\sigma^2 n} \right) dV(x) > C_4(\Lambda_{n,2} + L_{n,4});$$

for $B(t) = t^2 e^{-t^2/2}$ we obtain

$$|I| = n\sqrt{\pi} \left| \int \frac{x}{2\sigma\sqrt{n}} e^{-x^2/4\sigma^2 n} dV(x) \right| > \sqrt{\pi} \left(\frac{1}{8} |L_{n,3}| - \Lambda_{n,2} - L_{n,4} \right).$$

Hence it follows that

$$\Delta_n > C_5 \psi_{n,2} + O(1/n). \quad (7)$$

Under the conditions of Theorem 1, $n\psi_{n,2} \rightarrow \infty$ as $n \rightarrow \infty$, and, consequently, we have $\Delta_n > C_6 \psi_{n,2}$. From this (2) follows.

Suppose now that X_n has a lattice distribution with maximal span h . Putting in (6) $B(t) = \exp[-\frac{1}{2}(t + \frac{2\pi}{h}\sigma\sqrt{n})^2]$, we easily find that

$$\Delta_n \geq C_2 \left| \int_{|t| < T_n} \frac{f_n(t) e^{-t^2/2}}{t - \frac{2\pi}{h}\sigma\sqrt{n}} dt \right| + O\left(\psi_{n,2} + \frac{1}{\sqrt{n}}\right) > \frac{C_7}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

The last inequality and inequality (7) complete the proof of relation (3).

I express my sincere gratitude to my adviser V. V. Petrov for his constant attention to this work.

Leningrad State University
named after A. A. Zhdanov

Received
6 IV 1967

REFERENCES

- ¹ C. G. Essen, *Acta Math.*, **77**, 1 (1945).
- ² I. A. Ibragimov, *Theory of Probability and Its Applications*, **11**, 632 (1966).
- ³ V. V. Petrov, *Vestnik Leningrad. Univ.*, No. 19, 150 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.