
Soviet-era science, translated into English

FIXED POINTS OF NONCOMPACT MAPPINGS

V. D. Milman, P. D. Mil'man

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.77390>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.911+517.948+513.881

MATHEMATICS

V. D. MIL' MAN, P. D. MIL' MAN

FIXED POINTS OF NONCOMPACT MAPPINGS

(Presented by Academician L. V. Kantorovich, 11 III 1968)

1. In the present note we give a construction which originally arose as a new simple proof of the Schauder-Tikhonov principle. At the same time this construction made it possible to obtain a number of new existence theorems for fixed points (Theorems 1-6).

Let B be a Banach space and B^* its conjugate space. Denote by $D(B)$ the unit ball ($D(B) = \{x \in B : \|x\| \leq 1\}$). Let $K \subset D(B)$ be a convex closed set, and let a set $F \subset B^*$ separate the points of K (i.e., for $x_1, x_2 \in K$ and $x_1 \neq x_2$ there exists $f \in F$ such that $f(x_1) \neq f(x_2)$). Suppose that K is compact in the weak topology induced by the functionals from the linear set $F \subset B^*$ (the so-called $\sigma(F)$ -topology; we shall sometimes simply write weak topology). Here $\tilde{F} \subset F$. We shall be interested in conditions on a mapping $A : K \rightarrow K$ under which a fixed point exists.

2. **Lemma 1.** Let K_n be a convex compactum in E_n ($\dim E_n = n$); let A be a multivalued mapping of K_n , namely $Ax = C(x)$, where $C(x)$ is a convex closed set in K_n . Suppose the mapping has a closed graph (i.e., the closed set $\Gamma_A = \{(x, Ax)\}_{x \in K_n} \subset K_n \times K_n$). Then there exists $x \in K_n$ such that $x \in C(x)$.

The proof is carried out by reduction to the usual Brouwer principle.

3. **The basic construction.** Let $A : K \rightarrow K$ and suppose the conditions of item 1 are satisfied. For $F_n = \{f_i\}_{i=1}^n \subset F$ consider

$$P(F_n)x = (f_1(x), \dots, f_n(x)) = y^{(n)} \in E_n.$$

$P(F_n)$ is a mapping of K into n -dimensional space, and

$$P(F_n)K = K(F_n)$$

is a convex closed set in the space

$$B / \left\{ \bigcap_{i=1}^n (x : f_i(x) = 0) \right\}.$$

For $y \in K(F_n)$ denote the complete inverse image in K by

$$\mathfrak{X}_y = P^{-1}(F_n)y = \{x : P(F_n)x = y \text{ and } x \in K\}.$$

Note that if Y is a closed set in $K(F_n)$, then

$$\mathfrak{X}_Y = P^{-1}(F_n)Y$$

is a closed set in K in the $\sigma(F)$ -topology.

We now define a multivalued operator $\bar{A}(F_n)$, acting in $K(F_n)$, by the formula

$$\bar{A}(F_n)y_0 = \{y_\alpha \in K(F_n) : \exists x_\alpha \in \mathfrak{X}_{y_0} \text{ and } P(F_n)Ax_\alpha = y_\alpha\}.$$

In other words,

$$\bar{A}(F_n)y_0 = P(F_n)AP^{-1}(F_n)y_0.$$

From the operator $\bar{A}(F_n)$ we construct the operator $A(F_n)$:

$$A(F_n)y_0 = \text{conv}(\bar{A}(F_n)y_0),$$

where $\text{conv } M$ denotes the convex closed hull of the set M .

Requirement on the operator A :

(I). For every F'_m there exists $F_n \supset F'_m$ such that $A(F_n)$ is continuous* at every interior point of $K(F_n)$.

* A multivalued mapping $A : K \rightarrow K$ is continuous if, for every $x_0 \in K$ and $\varepsilon > 0$, there exists $\delta > 0$ such that from $\rho(x, x_0) < \delta$ it follows that

$$\rho(Ax, Ax_0) = \max \left\{ \sup_{y \in Ax} \rho(y, Ax_0); \sup_{y \in Ax_0} \rho(y, Ax) \right\} < \varepsilon.$$

Lemma 2. If the operator A is uniformly continuous on K , then (I) is satisfied.

The requirement of continuity of $A(F_n)$ only at the interior points of $K(F_n)$ is caused by the arbitrariness of K . In the case when A is weakly continuous (the Schauder-Tikhonov case), or $K = D(B)$ and is locally uniformly convex (see (3), p. 188), all $A(F_n)$ are continuous on all of $K(F_n)$. Below, for simplicity of exposition, precisely this is assumed; however, Proposition 1 is formulated in full generality.

By Lemma 1, from the continuity of $A(F_n)$ there follows the existence of $y_0 \in K(F_n)$ such that $y_0 = A(F_n)y_0$. Denote the set of such y by $N(A; F_n)$, and its complete inverse image in K , $P^{-1}(F_n)N = \mathfrak{N}(A; F_n)$. From the preceding it is clear that $\mathfrak{N}(A; F_n)$ is a closed set in K in the $\sigma(F)$ -topology (since $N(A; F_n)$ is closed in $K(F_n)$) and, as is easy to show, is a centered family with respect to $\{F_n\}$. Hence:

Proposition 1. If condition (I) on A is satisfied, then there exists

$$\emptyset \neq \bigcap_{F_n \subset F} \mathfrak{N}(A; F_n) = \mathfrak{N}(A) \subset K.$$

4. We shall now be interested in conditions on A under which $\mathfrak{N}(A)$ consists of fixed points of the mapping A . Let $x_0 \in \mathfrak{N}(A)$, $x_1 = Ax_0$. Suppose that $x_1 \neq x_0$. By the construction of $\mathfrak{N}(A)$, this means that for every F_n the set*

$$H_{F_n}(x_0) \equiv H_{F_n}(\bar{a}) = \{x \in K : f_i(x) = f_i(x_0) \text{ for } f_i \in F_n\},$$

where $\bar{a} = (f_1(x_0), \dots, f_n(x_0))$, is carried by the operator A into a set whose closed convex hull contains x_0 . Thus:

Proposition 2. If for the mapping $A : K \rightarrow K$ there is satisfied

$$(II) \text{ for every } x_0 \in K, \quad Ax_0 \neq x_0, \quad \exists F_n = \{f_i\}_{i=1}^n$$

such that

$$x_0 \notin \text{conv } AH_{F_n}(x_0),$$

then $\mathfrak{N}(A)$ coincides with the set of fixed points of the mapping A .

Remark 1. The Schauder-Tikhonov principle itself has in fact already been proved, since if A is continuous in the $\sigma(F)$ -topology, then, as shown above, $\mathfrak{N}(A) \neq \emptyset$, and (II) is easily verified.

Indeed, if $Ax_0 = x_1 \neq x_0$, then there exists a separating functional $f_0 \in F$, i.e. $f_0(x_0) \neq f_0(x_1)$. This means that x_1 has a neighborhood in the $\sigma(F)$ -topology not containing x_0 . Such a neighborhood is

$$H_{f_0, \varepsilon_0}(x_1) = \{x \in K : |f_0(x) - f_0(x_1)| \leq \varepsilon_0\}.$$

The complete inverse image of this neighborhood is a neighborhood of x_0 by the continuity of A in the $\sigma(F)$ -topology. Hence there exist $F_n = \{f_i\}_{i=1}^n$ and $\varepsilon > 0$ such that the neighborhood $V(x_0)$ of the point x_0

$$H_{F_n, \varepsilon}(x_0) = \{x \in K : |f_i(x) - f_i(x_0)| < \varepsilon \text{ for } i = 1, \dots, n\} \subset A^{-1}H_{f_0, \varepsilon_0}(x_1).$$

It follows that (II) is satisfied, since the image $H_{F_n, \varepsilon}$ is contained in the closed convex set $H_{f_0, \varepsilon_0}(x_1)$, which does not contain x_0 . Let us also note that, in a completely analogous way, one obtains a generalization of the Schauder-Tikhonov principle for multivalued mappings (Kakutani theorem, see (1, 2))—an infinite-dimensional analogue of Lemma 1.

In what follows we present various variants of requirements on A and K under which (II) is satisfied and, consequently, the mapping A has a fixed point. Uniform continuity of A on K (abbreviated u.c. A) will be assumed in the sequel.

5. First we give a proposition on superposition often used in examples.

* We introduce two notations for this set.

Proposition 3. If the mapping A_0 satisfies requirement (II), and A_1 and A_2 are continuous in the $\sigma(F)$ -topology, then the mapping $A_2A_0A_1$ satisfies requirement (II).

Theorem 1. Let the operator A_0 satisfy the following condition: there exists a set $F \subset F$ separating points of K , such that for every $f \in F$ and every a there is a b such that $A_0H_f(a) \subset H_f(b)$. Let A_1 and A_2 be continuous in the $\sigma(F)$ -topology and let $A_1A_0A_2$ be a r.n. operator $K \rightarrow K$. Then the mapping $A = A_1A_0A_2$ has a fixed point, and $\mathfrak{N}(A)$ is the set of all fixed points of A .*

Example-theorem 1a. Let $B(G)$ be the Banach space of functions defined on $G \subset E_n$, and suppose that for any $\alpha \in G$ the functionals $f_\alpha: f_\alpha(x(t)) = x(\alpha)$, where $t \in G$ and $x(t) \in B(G)$, are continuous. Denote $\{f_\alpha\}_{\alpha \in G} = \tilde{F}$. Let $K \subset D(B(G))$ and let it be weakly compact in the $\sigma(F)$ -topology, where $\tilde{F} \subset F$. Then, for any function $\varphi(t; u)$ continuous in u , the Hammerstein operator

$$Ax = \int_G K(s, t) \varphi(t; x(t)) dt$$

has a fixed point if and only if $A: K \rightarrow K$ and is uniformly continuous on K .

Theorem 1 was given by us for greater transparency. The more general Theorem 2 below also makes it possible to analyze the case of the Urysohn operator:

$$Ax = \int_G K(s, t; x(t)) dt.$$

Theorem 2. Suppose that for the mapping $A_0: K \rightarrow K$ there exists a set $F \subset F$ separating points of K , such that for every $x_0 \in K$, every $F_n \subset F$, and $\varepsilon > 0$ there are $F_m = \{f_i\}_{i=1}^m \supset F_n$ and $b = (b_1, \dots, b_m)$ such that

$$A_0H_{F_m}(x_0) \subset H_{F_m, \varepsilon}(b).$$

Then for any mappings A_i ($i = 1, 2$) continuous in the weak topology and such that $A_2A_0A_1: K \rightarrow K$ and r.n. on K , the set $\mathfrak{N}(A_2A_0A_1) \neq \emptyset$ and coincides with the set of fixed points of the mapping $A_2A_0A_1$.*

Theorem 3. Let the linear span F be dense in B^* (this is always fulfilled, for example, for reflexive spaces). Consider r.n. mappings $A: K \rightarrow K$, where

$$A = \sum_{i=1}^n A_2^{(i)} A_0^{(i)} A_1^{(i)}.$$

If each summand is continuous in B and satisfies the conditions of Theorem 2 (but does not necessarily carry $K \rightarrow K$), then $\mathfrak{N}(A) \neq \emptyset$ and coincides with the set of fixed points of the mapping A .

6. Asymptotically weakly continuous and asymptotically affine mappings. We shall call a mapping A **asymptotically weakly continuous** if, for every $x \in K$ and $\varepsilon > 0$, there is an $E^{N(x,\varepsilon)}$ ($\text{codim } E^N = N$) such that the operator A , considered on $(x + E^N) \cap K$, is an ε -perturbation of a weakly continuous one, i.e., for

$$M = A[(x + E^N) \cap K]$$

and every $U(Ax) \in \sigma$ there is a $V(x) \in \sigma$ such that**

$$AV(x) \subset M_\varepsilon \cap U(Ax).$$

We shall call a mapping $A: K \rightarrow K$ **asymptotically affine** if, for every $x \in K$ and $\varepsilon > 0$, there is an $E^{N(x,\varepsilon)}$ ($\text{codim } E^N = N$) such that

$$\left\| A \left(\frac{x_1 + x_2}{2} \right) - \frac{Ax_1 + Ax_2}{2} \right\| < \varepsilon \quad \text{for } x_i \in (x + E^N) \cap K \quad (i = 1, 2).$$

Theorem 4. For the classes of mappings indicated in this section, requirement (II) is fulfilled and, consequently, under the condition of uniform continuity of these mappings they possess fixed points.

* A more general assertion is true, with A_0, A_1 , and A_2 acting in different vector spaces.

** M_ε denotes the ε -enlargement of the set M .

7. **Theorem 5.** If a u.c. mapping $A: K \rightarrow K$ satisfies the condition: for any $x_0 \in K$, $f \in F$, and $\varepsilon > 0$ there is an $F_n \subset B^*$, $f \in F_n$, such that $d(AH_{F_n}(x_0)) < \varepsilon$, where $d(M)$ is the diameter of the set M , then $\emptyset \neq \mathfrak{N}(A)$ coincides with the fixed points of the mapping A .

8. **On one class of mappings $K \rightarrow K$.** Instead of the sets $\mathfrak{N}(A; F_n)$, introduced in Sec. 3 of the present paper, consider their closed ε -neighborhoods $\mathfrak{N}_\varepsilon(A; F_n)$. Denote

$$\mathfrak{N}_\varepsilon(A) = \bigcap_{(F_n) \subset F} \mathfrak{N}_\varepsilon(A; F_n)$$

and

$$\rho_A(M) = \sup\{\|Ax - x\| \text{ for } x \in M\}.$$

Definition of the class $\mathfrak{A}$. $A \in \mathfrak{A}$ if and only if $\rho(\mathfrak{N}_\varepsilon(A)) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and A is uniformly continuous.

Obviously, if $A \in \mathfrak{A}$, then $\mathfrak{N}(A)$ coincides with the set of fixed points of the mapping A . Note that in all the examples of mappings considered above (Theorems 1–5), the operators belonged to the class $\mathfrak{A}$. Moreover:

Theorem 6. Any finite superposition of the operators indicated in Theorems 1–5 is also an operator from $\mathfrak{A}$.

However, the introduction of the class $\mathfrak{A}$ is prompted by the following theorem.

Theorem 7. The class $\mathfrak{A}$ is closed in the uniform operator topology.

Combining this theorem with Theorem 3 and with Example-Theorem 1a, one can obtain fixed-point theorems for a broad class of Urysohn operators.

Institute of Chemical Physics
Academy of Sciences of the USSR

Received
26 II 1968

CITED LITERATURE

¹ S. Kakutani, Duke Math. J., **8**, No. 3 (1941). ² S. Kakutani, J. Math. Soc. Japan, **3**, No. 1 (1951). ³ M. M. Day, *Normed Linear Spaces*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.