

θ -PROXIMITIES AND θ -ABSOLUTES

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Abstract

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MATHEMATICS

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θ -PROXIMITIES AND θ -ABSOLUTES

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Introduction. This article is closely connected with the paper ⁽⁷⁾, almost all of whose results are contained here in a more general form. The concept of θ -proximity, introduced in ⁽⁷⁾, is extended here to the case of all Hausdorff spaces. It turns out that, with the aid of θ -proximity, one can describe all irreducible perfect and even θ -perfect (continuity is replaced by θ -continuity in the sense of S. V. Fomin ⁽⁹⁾) mappings of completely regular spaces. If proximities on a given space correspond, as Yu. M. Smirnov showed ⁽⁶⁾, to bicomact extensions of this space (bicomact extensions of homeomorphic preimages of this space), then θ -proximities on a given space correspond to bicomact extensions of completely regular preimages of this space under θ -perfect irreducible mappings.

The principal method of the work is the method of centered systems of open sets. This method was first considered by P. S. Aleksandrov ⁽¹⁾, who, with its aid, gave a very transparent construction of the maximal bicomact extension of a completely regular space. Yu. M. Smirnov ⁽⁶⁾, generalizing P. S. Aleksandrov's method, used the concept of proximity to construct all bicomact extensions of completely regular spaces. In an entirely different direction, P. S. Aleksandrov's method was generalized in the works of S. V. Fomin and S. Iliadis (see ^(2,3,8)), in which, in particular, the maximal bicomact extension of a maximal θ -perfect irreducible preimage of a given Hausdorff space was constructed. In the present work the method of centered systems of open sets, with the aid of the concept of θ -proximity, makes it possible to obtain a result of which all those listed above are special cases.

The concept of a θ -proximally continuous mapping is introduced. In this connection the θ -mappings introduced in ⁽⁷⁾ are perfect irreducible θ -proximally continuous mappings.

A result analogous to Yu. M. Smirnov's theorem on the extension of δ -continuous mappings to the corresponding bicomact extensions of proximity spaces is proved. It is shown that in every class of θ -spaces mapped onto one another by means of multivalued regular θ -mappings there exists a projective space (a θ -absolute). At the same time, V. I. Ponomarev's theorem on the absolute of a topological space can be formulated in terms of θ -proximity. In this formulation it turns out to be a special case of the theorem on the θ -absolute. The concept of

θ -extremal disconnectedness, generalizing the concept of extremal disconnectedness, is also introduced. There is a natural connection between θ -absolutes and θ -extremally disconnected θ -spaces. The θ -absolutes are precisely the ordinary proximity spaces, and only they. From this point of view, ordinary proximities are distinguished among all θ -proximities in the same way as extremally disconnected spaces are distinguished among all Hausdorff spaces.

Definitions and statements of theorems. We shall say that a θ -proximity is given on a topological space X if, for any two subsets A and B of X , one of the following two conditions holds:

relations: $A\theta B$ (A is θ -near to B) or $A\bar{\theta}B$ (A is θ -far from B), with the following axioms satisfied:

I. $A\theta B \Rightarrow B\theta A$.

II. $A\theta B_i, i = 1, 2 \iff A\bar{\theta}(B_1 \cup B_2)$.

III. $\emptyset\bar{\theta}X$.

IV. $\{x\}\theta\{y\} \Rightarrow x = y$.

V. $A\bar{\theta}B \Rightarrow$ there exists a canonically open set C such that $\bar{C}\bar{\theta}B$ and $A\bar{\theta}(X \setminus [C])$.

VI. $\{x\}\theta A \Rightarrow$ any neighborhoods of the point x and of the set A intersect.*

It follows from the axioms that θ -nearnesses exist only on Hausdorff spaces. It turns out (see the corollary to Theorem 1) that on every Hausdorff space there exist θ -nearnesses. The most general example of a θ -nearness is given by

Theorem 1. Let $f : Z \rightarrow X$ be a θ -perfect irreducible mapping of a completely regular space Z onto the space X , and let bZ be a bicomact extension of the space Z . Then bZ induces on X the following θ -nearness:

$$A\bar{\theta}B \iff [f^{-1}A]_{bZ} \cap [f^{-1}B]_{bZ} = \emptyset.$$

Corollary. On every Hausdorff space there exists a θ -nearness.

This follows from the fact that every Hausdorff space has an absolute which is a completely regular space.

Theorem 2. Every θ -nearness on a Hausdorff space X determines a completely regular space X_θ , a θ -perfect irreducible mapping $\pi_{X_\theta} : X_\theta \rightarrow X$ onto X , and a bicomact extension $b_\theta X_\theta$, which induces (in the sense of Theorem 1) the given θ -nearness.

Theorem 3. Let $f_1 : Z_1 \rightarrow X$ and $f_2 : Z_2 \rightarrow X$ be θ -perfect irreducible mappings of completely regular spaces Z_1 and Z_2 onto X . If the bicomact extensions $b_1 Z_1$ and $b_2 Z_2$ induce on X the same θ -nearness, then there exists a homeomorphism $g : b_1 Z_1 \rightarrow b_2 Z_2$ such that $gZ_1 = Z_2$ and $f_1 = f_2 g$.

Corollary 1. The set of θ -nearnesses on a bicompactum is in one-to-one correspondence with the set of perfect irreducible mappings onto this bicompactum.**

Corollary 2. Every θ -nearness on an extremally disconnected space is an ordinary nearness.

Corollary 3. If a θ -nearness on the space X is an ordinary nearness, then the mapping $\pi_{X_\theta} : X_\theta \rightarrow X$ is a homeomorphism, and the bicompactum $b_\theta X_\theta$ is a bicompact extension of the proximity space X .

Let X and Y be two θ -spaces, i.e., two topological spaces on each of which a θ -nearness is given. A mapping $f : X \rightarrow Y$ will be called **θ -nearness-continuous** if two conditions are satisfied:

$$1^\circ. A, B \subset Y, A\bar{\theta}B \Rightarrow f^{-1}A\bar{\theta}f^{-1}B.$$

$$2^\circ. A, B \subset Y, A\theta B \Rightarrow \langle f^{-1}[A] \rangle \theta \langle f^{-1}[B] \rangle.***$$

* The first four axioms are the axioms of a generalized proximity space. Axiom V is a weakening of the normality axiom for a proximity space, compatible with a topological space. Axiom VI shows the connection of θ -nearness with topology. For regular spaces axiom VI is equivalent to axiom VI': $\{x\}\theta A \Rightarrow x \in [A]$ (axiom IV of paper (7)).

** Two mappings $f_1 : Z_1 \rightarrow X$ and $f_2 : Z_2 \rightarrow X$ are identified if there exists a homeomorphism $g : Z_1 \rightarrow Z_2$ such that $f_1 = f_2 g$.

*** $\langle C \rangle$ denotes the interior of the set C .

Lemma. Every θ -proximity continuous mapping is θ -continuous.

We shall call a θ -space X a **regular θ -space** if the space X is a regular space.

Theorem 4. Let $f : X \rightarrow Y$ be a θ -proximity continuous mapping of a θ -space X onto a regular θ -space Y . Then there exists a continuous mapping $f_\theta : b_\theta X_\theta \rightarrow b_\theta Y_\theta$ of the bicompactum $b_\theta X_\theta$ onto the bicompactum $b_\theta Y_\theta$ such that $f_\theta X_\theta \subset Y_\theta$ and the diagram is commutative

$$\begin{array}{ccc} X_\theta & \xrightarrow{f'_\theta} & Y_\theta \\ \pi_{X_\theta} \downarrow & & \downarrow \pi_{Y_\theta} \\ X & \xrightarrow{f} & Y \end{array}$$

where f'_θ is the restriction of the mapping f_θ to X_θ .

Remark. The restriction of the mapping f_θ to X_θ is not, generally speaking, a mapping onto Y_θ , but is only a mapping onto some dense subset of the space Y_θ .

Theorem 5. If a θ -proximity continuous mapping $X \rightarrow Y$ of a θ -space X onto a θ -space Y is perfect, then $f_\theta X_\theta = Y_\theta$.

It has been possible to get rid of the condition of regularity of the θ -space Y only by imposing restrictions on the mapping f .

Theorem 6. Let $f : X \rightarrow Y$ be a θ -proximity continuous mapping of a θ -space X onto a θ -space Y . Suppose that the mapping f is closed and irreducible. Then there exists a continuous irreducible mapping f'_θ of the bicomactum $b_\theta X_\theta$ onto the bicomactum $b_\theta Y_\theta$ such that $f'_\theta X_\theta \subset Y_\theta$ and the diagram is commutative

$$\begin{array}{ccc} X_\theta & \xrightarrow{f'_\theta} & Y_\theta \\ \pi_{X_\theta} \downarrow & & \downarrow \pi_{Y_\theta} \\ X & \xrightarrow{f} & Y \end{array}$$

If, moreover, the mapping f is bicomact, then $f^{-1}Y_\theta = X_\theta$, and the mapping f'_θ is perfect and irreducible.

Corollary. Every multivalued θ -mapping* $f : X \leftarrow Z \rightarrow Y$ of a θ -space X onto a θ -space Y generates such a multivalued irreducible mapping $f_\theta : b_\theta X_\theta \leftarrow b_\theta Z_\theta \rightarrow b_\theta Y_\theta$ that $\varphi^{-1}X_\theta = Z_\theta = \psi^{-1}Y_\theta$ and $f\pi_{X_\theta} = \pi_{Y_\theta}f'_\theta$.

A θ -proximity continuous mapping $f : X \rightarrow Y$ will be called a **θ -mapping** if f is θ -perfect and irreducible.** In accordance with (4), a θ -mapping $f : X \rightarrow Y$ will be called a **regular θ -mapping** if $A, B \subset Y$, $A\theta B \Rightarrow f^{-1}A\theta f^{-1}B$.

Definition. A θ -space X^θ is called a **θ -absolute** of the θ -space X if there exists a canonical projection $\pi_X : X^\theta \rightarrow X$, which is a regular θ -mapping, such that for every regular θ -mapping $f : Z \rightarrow X$ onto X there exists a regular θ -mapping $g : X^\theta \rightarrow Z$ onto Z such that $\pi_{X^\theta} = fg$.

* A multivalued mapping $f : X \rightarrow Y$ of θ -spaces will be called a **multivalued θ -mapping** if there exist a θ -space Z and such closed, irreducible, bicomact and θ -proximity continuous mappings $f_X : Z \rightarrow X$, $f_Y : Z \rightarrow Y$, that $f = f_Y f_X^{-1}$. For regular spaces this definition coincides with the definition given in (7).

** For regular spaces this definition coincides with the definition given in (7).

A space X endowed with the maximal θ -nearness θ_a (two sets are θ_a -far if and only if they have disjoint neighborhoods) will be called a **maximal θ -space**. It can be shown that every θ -perfect irreducible mapping $f : X \rightarrow Y$ of a maximal θ -space X onto a maximal θ -space Y is a regular θ -mapping. After this, the theorem of V. I. Ponomarev ⁽⁵⁾ on the absolute of a topological space can be reformulated as follows:

Every maximal θ -space X has an absolute X^θ with respect to the class of regular θ -mappings of maximal θ -spaces.

The following theorem is a generalization of V. I. Ponomarev' s theorem.

Theorem 7. Every θ -space X has a θ -absolute X^θ , which is unique in the sense that any two θ -absolutes X_1^θ and X_2^θ of the space X are connected by a θ -equimorphism $h : X_1^\theta \rightarrow X_2^\theta$, satisfying the condition $\pi_{X_2^\theta} h = \pi_{X_1^\theta}$. For every multivalued regular θ -mapping $f : X \rightarrow Y$ onto Y there exists a θ -equimorphism $f^\theta : X^\theta \rightarrow Y^\theta$ such that $f = \pi_{Y^\theta} f^\theta \pi_{X^\theta}^{-1}$. Every regular θ -mapping onto a θ -absolute is a θ -equimorphism.

Corollary 1. In order that a θ -space X be a θ -absolute, it is necessary and sufficient that the θ -nearness on X be a nearness.

As is known, a space is extremally disconnected if and only if, for any two disjoint open subsets of it, their closures do not intersect. In terms of θ -nearness this means that any two θ -far sets in the maximal θ -nearness have θ -far closures. What has been said above justifies the following

Definition. A θ -space X is called **θ -extremally disconnected** if, in it, any two θ -far sets have θ -far closures.

Corollary 2. A θ -space X is a θ -absolute if and only if it is θ -extremally disconnected.

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* A mapping $f : X \rightarrow Y$ of a θ -space X onto a θ -space Y is called a **θ -equimorphism** if it is one-to-one and θ -nearness-continuous in both directions.

Note: Figure translations are in progress. See original paper for figures.

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