

SPHERICAL SOUND WAVES IN RIEMANNIAN SPACE

K. P. Stanyukovich

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.76410>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 530.12

HYDROMECHANICS

K. P. STANYUKOVICH

SPHERICAL SOUND WAVES IN RIEMANNIAN SPACE

(Presented by Academician L. I. Sedov on 23 XI 1967)

1. Basic equations for spherical waves. The equations describing spherically symmetric motions of a medium in its own gravitational field have the form (1)

$$\frac{1}{c\theta^2}[Au_t + uu_r] - \frac{\omega^2}{c^2} \left[(\ln V)_r + \frac{Au}{c^2}(\ln V)_t \right] + \frac{1}{2} \left[\nu_r + \frac{Au}{c^2}\lambda_t \right] = \frac{\theta^2 T^0 \sigma_r}{W}; \quad (1,1)$$

$$- [A(\ln V)_t + u(\ln V)_r] + \frac{1}{\theta^2} \left[u_r + \frac{Au}{c^2}u_t \right] + \frac{1}{2} [A\lambda_t + u\nu_r] = 0; \quad (1,2)$$

$$A\sigma_t + u\sigma_r = 0; \quad (1,3)$$

$$\nu_r = \frac{u^2}{c^2} \left[\lambda_r + \frac{e^\lambda - 1}{2} + \kappa pre^\lambda \right] + \frac{e^\lambda - 1}{2} + \kappa pre^\lambda \quad (1,4)$$

or

$$A\lambda_t + u\lambda_r + u \left[\frac{e^\lambda - 1}{2} + \kappa pre^\lambda \right] = 0; \quad (1,4a)$$

$$A(1 + u^2/c^2)\lambda_t + u(\lambda + \nu)_r = 0 \quad (1,5)$$

or

$$(re^{-\lambda})_r = 1 - \frac{\kappa r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2} \right). \quad (1,5a)$$

This system of 5 independent equations uniquely determines u, V, σ, λ and ν for a given equation of state $p = p(V, \sigma)$ and with use of the identity

$\partial(p, V)/\partial(T^0, \sigma) = 1$; here p is the pressure; T^0 is the temperature; V is the specific volume; σ is the entropy; $u = (u_\alpha u^\alpha)^{1/2}$ is the 3-velocity; $\theta^2 = 1 - u^2/c^2$; $\omega^2/c^2 = -(\partial \ln W / \partial \ln V)_\sigma$ is the speed of sound; $W = pV + E = pV + \rho V c^2$ is the heat content; $\rho c^2 = \varepsilon$ is the energy density, with $A = e^{(\lambda-\nu)/2}$, $dE = d(\rho V c^2) = T^0 d\sigma - p dV$, where λ and σ determine the metric of the centrally symmetric field:

$$-ds^2 = -c^2 dt^2 e^\nu + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (1,6)$$

In the case of equilibrium, when $u = 0$, we shall have:

$$V p_r / W + \nu_r / 2 = 0; \quad (1,7)$$

$$\lambda_t = 0, \quad \sigma_t = 0, \quad p_t = 0;$$

$$\nu_r = (e^\lambda - 1)/r + \kappa p r e^\lambda; \quad (1,8)$$

$$d(re^{-\lambda})/dr = 1 - \kappa r^2 \varepsilon. \quad (1,9)$$

Eliminating λ and ν from (1,7), (1,8), and (1,9), we arrive at the well-known Oppenheimer-Volkoff equation (2)

$$\frac{d}{dr} \left[\frac{r(1 + \kappa r^2 p)}{1 - r dp/(p + \varepsilon) dr} \right] = 1 - \kappa r^2 \varepsilon. \quad (1,10)$$

We shall take the parameters determined by these equations as the zeroth approximation. For $u \neq 0$, when $u/c \ll 1$, we shall have:

$$A u_t - \omega^2 (\ln V)_r + \frac{c^2}{2} \left[\nu_r + \frac{A u}{c^2} \lambda_t \right] = \frac{T^0 \sigma_r}{W}; \quad (1,11)$$

$$-A (\ln V)_t + u_r + 2u/r + \frac{1}{2} [A \lambda_t + u \nu_r] = 0; \quad (1,12)$$

$$A \sigma_t + u \sigma_r = 0; \quad (1,13)$$

$$\nu_r = (e^\lambda - 1)/r + \kappa p r e^\lambda; \quad (1,14)$$

$$A \lambda_t + u(\lambda + \nu)_r = 0; \quad (1,15)$$

or

$$\partial(re^{-\lambda})/\partial r = 1 - \kappa r^2 \varepsilon. \quad (1,15a)$$

2. Sound waves in Einstein space. Consider the simplest case $p_0 = \text{const}$, then $\nu_r = 0$, whence, without loss of generality, one may put $\nu_0 = 0$; in this case $e^{-\lambda_0} = 1 + \kappa r^2 p_0$, or

$$\lambda_0 = -\ln(1 + \kappa r^2 p_0), \quad \sigma_0 = \text{const}, \quad \partial \lambda_0 / \partial r = -2\kappa r p_0 / (1 + \kappa r^2 p_0). \quad (2,1)$$

Further we shall have

$$(1 + \kappa r^2 p_0) \left(1 + \frac{2\kappa p_0 r^2}{1 + \kappa r^2 p_0} \right) = 1 + 3\kappa r^2 p_0 = 1 - \kappa r^2 \varepsilon_0, \quad (2,2)$$

which gives the equation of state

$$3p_0 + \varepsilon_0 = 0, \quad (2,3)$$

which is the equation of state of the spherical Einstein world. Here $p_0 = p_m + p_f$; $\varepsilon_0 = \varepsilon_m + \varepsilon_f$; $\varepsilon_f = 3p_f$, where the indices f and m refer to the values of ε and p for the field and matter, respectively. In this case $\varepsilon_f = -(3p_m + \varepsilon_m)/2$, $\varepsilon_0 = (\varepsilon_m - 3p_m)/2$. Obviously, ε_f should be understood as the density of the negative energy of the gravitational field itself. In this case there is no need to introduce the so-called λ -term into the Einstein equations. Instead, the energy and pressure of the field are introduced. Further, it is evident that $\kappa \varepsilon_0 = 3/a^2$; $\kappa p_0 = -1/a^2$;

$$1 + \kappa p_0 r^2 = 1 - r^2/a^2 = e^{-\lambda}; \quad (2,4)$$

$$-ds^2 = -c^2 dt^2 + \frac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (2,5)$$

i.e. the metric of the Einstein world.

For $u/c < 1$ we shall have $\lambda = \lambda_0 + \Delta\lambda = -\ln(1 + \kappa r^2 p_0) + \Delta\lambda$, $\nu = \Delta\nu$, $V = V_0 + \Delta V$, $p = p_0 + \Delta p$, $\varepsilon = \varepsilon_0 + \Delta\varepsilon$, $\sigma = \sigma_0 + \Delta\sigma$. (If $\lambda = -\nu$, $e^\nu = 1 + \kappa r^2 p_0/3$, then we obtain that $\varepsilon_0 + p_0 = 0$; this equation of state corresponds to the de Sitter metric.)

$$-ds^2 = -c^2 dt^2 \left(1 - \frac{r^2}{a^2} \right) + \frac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2,6)$$

The investigation of small perturbations for the de Sitter world is analogous to what we do for the Einstein world.

The basic equations (1,11), (1,13) take the form

$$Au_t - \omega_0^2 \frac{\Delta V_r}{V_0} + \frac{c^2}{2} \Delta \nu_r = \frac{T^0}{W} \sigma_r, \quad A \frac{\Delta V_t}{V_0} + u_r + \frac{2u}{r} + \frac{A \Delta \lambda_t}{2} = 0, \quad (2,7)$$

$$\Delta \sigma_t = 0,$$

i.e.

$$\Delta \sigma = \Delta \sigma(r). \quad (2,8)$$

In this case

$$A = e^{(\lambda_0 + \Delta \lambda - \Delta \nu)/2} = \frac{1}{\sqrt{1 - r^2/a_0^2}} \left[1 + \frac{\Delta \lambda - \Delta \nu}{2} \right]. \quad (2,9)$$

For the time being let us consider the more particular case when $\Delta \sigma = 0$, i.e. purely isentropic sound waves.

Equations (2,6) and (2,7) then take the form

$$\frac{u_t}{\sqrt{1 - r^2/a^2}} - \frac{\omega_0^2}{V_0} (\Delta V)_r + \frac{c^2}{2} \Delta \nu_r = 0; \quad (2,10)$$

$$-\frac{(\Delta \nu)_t}{\sqrt{1 - r^2/a^2}} + u_r + \frac{2u}{r} + \frac{\Delta \lambda_t}{2\sqrt{1 - r^2/a^2}} = 0. \quad (2,11)$$

Further, equation (1,14) gives

$$\nu_r = \frac{\Delta \lambda}{r} + \frac{\kappa r \Delta p}{1 - r^2/a^2} = \frac{\Delta \lambda}{r} - \frac{r \Delta p}{a^2 p_0 (1 - r^2/a^2)}, \quad (2,12)$$

and equation (1.15), which we shall write in the form

$$\frac{\Delta \lambda_t}{\sqrt{1 - r^2/a^2}} = \frac{2\chi r p_0 u}{1 - r^2/a^2},$$

immediately determines

$$\Delta\lambda_t = -\frac{2ru}{a^2\sqrt{1-r^2/a^2}}. \quad (2.13)$$

Now equations (2.10) and (2.11) can be written in the form

$$\frac{u_t}{\sqrt{1-r^2/a^2}} - \frac{\omega_0^2}{v_0}(\Delta v)_r + \frac{c^2}{2} \left[\frac{\Delta\lambda}{r} - \frac{r\Delta p}{a^2 p_0} \left(1 - \frac{r^2}{a^2} \right) \right] = 0, \quad (2.14)$$

$$-\frac{(\Delta v)_t}{v_0\sqrt{1-r^2/a^2}} + u_r + \frac{2u}{r} - \frac{u_r}{a^2(1-r^2/a^2)} = 0. \quad (2.15)$$

Since

$$\frac{\omega_0}{V_0}\Delta v = V_0\Delta p,$$

we finally write equations (2.14) and (2.15) in the form

$$\frac{u_t}{\sqrt{1-r^2/a^2}} + v_0\Delta p_r + \frac{c^2}{2} \left[\frac{\Delta\lambda}{r} - \frac{r\Delta p}{a^2 p_0(1-r^2/a^2)} \right] = 0; \quad (2.16)$$

$$-\frac{v_0\Delta p_t}{\omega_0^2\sqrt{1-r^2/a^2}} + u_r + \frac{2u}{r} - \frac{ur}{a^2(1-r^2/a^2)} = 0. \quad (2.17)$$

Let $u = \overline{W}f(r)$, where $\overline{W} = \overline{W}(r, t)$; then (2.17) takes the form

$$-\frac{v_0\Delta p_t}{\omega_0^2\sqrt{1-r^2/a^2}} + f \left[\overline{W}_r + \overline{W} \left(\frac{f'}{f} + \frac{2}{r} - \frac{r}{a^2(1-r^2/a^2)} \right) \right] = 0;$$

we set the expression in parentheses equal to zero and determine

$$\frac{df}{f} = -\frac{2dr}{r} + \frac{rdr}{a^2(1-r^2/a^2)}, \quad (2.18)$$

whence

$$fr^2\sqrt{1-r^2/a^2} = B = \text{const}, \quad u = \frac{B\overline{W}}{r^2\sqrt{1-r^2/a^2}}.$$

In this case equation (2.17) can be written in the form

$$r^2V_0\Delta p_t/B\omega_0^2 + \overline{W}_r = 0, \quad (2.19)$$

whence

$$\Delta p = -\frac{B\omega_0^2}{r^2 v_0} \frac{\partial \psi}{\partial r}; \quad (2.20)$$

$$\overline{W} = \frac{\partial \psi}{\partial t} = \frac{ur^2}{B} \sqrt{1 - r^2/a^2} \quad \text{or} \quad u = \frac{B}{r^2 \sqrt{1 - r^2/a^2}} \frac{\partial \psi}{\partial t}. \quad (2.21)$$

In this case equation (2.16) takes the form

$$\begin{aligned} \frac{\partial^2 \psi}{\partial t^2} - \left(1 - \frac{r^2}{a^2}\right) \omega_0^2 \frac{\partial^2 \psi}{\partial t^2} + 2 \left(1 - \frac{r^2}{a^2}\right) \frac{\omega_0^2}{r} \frac{\partial \psi}{\partial r} + \\ + \frac{c^2}{2} \left[\frac{(1 - r^2/a^2)}{B} (\Delta \lambda)_r + \frac{r\omega_0^2}{a^2 p_0 v_0} \frac{\partial \psi}{\partial r} \right] = 0. \end{aligned} \quad (2.22)$$

From (2.13) we have

$$\sqrt{1 - r^2/a^2} \frac{\partial \Delta \lambda}{\partial t} = -\frac{2ur}{a^2} = -\frac{B}{a^2 r \sqrt{1 - r^2/a^2}} \frac{\partial \psi}{\partial t},$$

whence

$$\Delta \lambda = -\frac{B\psi}{(1 - r^2/a^2)ra^2} + \Phi^*(r).$$

Since for $\psi = 0$, $\Delta \lambda = 0$, it follows that $\Phi^*(r) = 0$, and finally

$$\Delta \lambda = -\frac{B\psi}{(1 - r^2/a^2)ra^2}. \quad (2.23)$$

Now equation (2.22) can be written in the form:

$$\psi_{x^0 x^0} + \frac{1}{2} \left[\psi_r \frac{r}{a^2} \frac{\omega_0^2}{p_0 V_0} - \frac{\psi}{a^2} \right] = \left(1 - \frac{r^2}{a^2}\right) \frac{\omega_0^2}{c^2} \left[\psi_{rr} - \frac{2\psi_r}{r} \right], \quad (2.24)$$

where $x^0 = ct$. We shall assume that the change in pressure and energy density occurs according to the laws of an ultrarelativistic gas; then

$$\Delta p = -\Delta \varepsilon/3, \quad \omega_0^2/c^2 = \Delta p/\Delta \varepsilon = -1/3, \quad -p_0 v_0 = c^2/3 = \omega_0^2;$$

in this case we shall have

$$\psi_{x^0 x^0} - \frac{1}{2a^2}[-r\psi_r + \psi] + \frac{1}{3}\left[1 - \frac{r^2}{a^2}\right]\left[\psi_{rr} - \frac{2\psi_r}{r}\right] = 0. \quad (2.25)$$

If we introduce a new function F and a new independent variable χ by means of the relation

$$\psi = a \sin \chi \exp \left[ikx^0 + \int \left(F - \frac{5}{4} \operatorname{tg} \chi \right) d\chi \right], \quad (2.26)$$

where $r = a \sin \chi$, then we arrive at the Riccati equation

$$F_\chi + F^2 = -\frac{5}{4 \cos^2 \chi} + 2 \operatorname{ctg}^2 \chi + 3k^2 a^2 + \frac{1}{16} \operatorname{tg}^2 \chi + 3. \quad (2.27)$$

A numerical solution of this equation presents no difficulties.

Analysis of the basic equation (2.24) leads to interesting results. At the center, for $r = 0$,

$$\psi_{x^0 x^0} \simeq \frac{\omega_0^2}{c^2} \left(-\frac{2\psi_r}{r} \right) = \frac{2}{3} \frac{\psi_r}{r}. \quad (2.28)$$

At the “periphery,” as $r \rightarrow a$,

$$\psi_{x^0 x^0} \simeq \frac{1}{2a^2}(a\psi_r + \psi). \quad (2.29)$$

In the central regions the amplitude Δp and the velocity value for the diverging wave decrease as r increases and increase for the converging wave; at the “periphery,” on the contrary, the amplitude of the converging wave decreases upon convergence, and the amplitude of the diverging wave increases as r tends to a .

Received
12 XI 1967

CITED LITERATURE

1. S. M. Kolesnikov, K. P. Stanyukovich, PMM, **29**, no. 4, 716 (1965).
2. J. Oppenheimer, G. Volkoff, Phys. Rev., **55**, 374 (1939).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.