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Abstract

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MATHEMATICS

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ON A BOUNDARY VALUE PROBLEM FOR LINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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In the present article the first boundary value problem is considered for a linear equation of parabolic type

$$Lu \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} (a_{ij} u_{x_j} + a_i u + f_i) + b_i u_{x_i} + au + f = 0 \quad (1)$$

with unbounded lower-order coefficients and free terms in the cylinder $Q_T = D \times [0 \leq t \leq T]$ ($T < \infty$), whose base is a bounded domain D of the Euclidean space E^n .

We take the initial and boundary conditions in the form

$$u|_{t=0} = \psi_0(x), \quad u|_{S_T} = 0. \quad (2)$$

Numerous works have been devoted to the study of existence and uniqueness of the generalized solution of problem (1), (2) in one class or another (see the detailed bibliography in ⁽¹⁻⁵⁾). In the monograph ⁽¹⁾ the latest results in this direction are set forth in detail. In the case when a_i^2, b_i^2, a with respect to the variable x belong to $L_p, p > n$, in ⁽¹⁾ conditions are given ensuring uniqueness and existence of a generalized solution from V_2 of problem (1), (2); here the usual embedding theorems are used essentially.

However, in the case $p < n$ these conditions are insufficient for the existence and uniqueness of a generalized solution from V_2 of problem (1), (2). In this case, some additional conditions are needed for unique solvability. Moreover, in the case $p < n$ the usual theorems do not work.

In the present article, in the case $1 < p < n$, in terms of spaces with mixed norm, conditions on the data of the problem are found under which the unique solvability of problem (1), (2) in V_2 holds. In addition, the smoothness of the

solution $u(x, t)$ with respect to t is studied (for the definitions of the spaces V_2 , $V_2^{1,0}$, V_2^* , see ⁽¹⁾).

Let s be a fixed natural number not exceeding n ; E^n is the n -dimensional Euclidean space of points $\bar{x}$; E^{n+1} is the $(n+1)$ -dimensional Euclidean space of points $(\bar{x}, t)$; $E^s(E^{n-s})$ is the s -dimensional $((n-s)$ -dimensional) space of points $\bar{x}_s(\bar{x}_{n-s})$, where $\bar{x} = (\bar{x}_s, \bar{x}_{n-s})$, $\bar{x}_s = (x_1, x_2, \dots, x_s)$, $\bar{x}_{n-s} = (x_{s+1}, x_{s+2}, \dots, x_n)$; D is a bounded domain of class $C(\bar{h}, \omega)$ in E^n (for the definition of the class $C(\bar{h}, \omega)$, see ⁽²⁾); Q_T is the cylinder $D \times [0 \leq t \leq T]$; $D_1 = D \cap (\bar{x}_{n-s} = \text{const})$; $D_2 = \text{pr}_{E^{n-s}} D$; $L_{(p_1, p_2, l)}(Q_T)$ is the Banach space consisting of all functions measurable on Q_T and having finite norm

$$\|f\|_{L_{(p_1, p_2, l)}(Q_T)} = \|\|f\|_{L_{(p_1, p_2)}(D)}\|_{L_l(0 \leq t \leq T)}.$$

Let $\Omega^{(\varepsilon)}$ denote the following set of points (r_1, r_2) of the plane $r_1 O r_2$:

$$\Omega_k^{(\varepsilon)} \equiv \left\{ \begin{array}{l} (r_1, r_2) : \quad k r_1 r_2 - (n-s)r_1 - s r_2 - \varepsilon = 0; \\ \infty > r_1 > \begin{cases} 1, & \text{if } s < k, \\ s/k, & \text{if } s \geq k; \end{cases} \\ \infty < r_2 > \begin{cases} 1, & \text{if } n-s < k, \\ (n-s)/k, & \text{if } n-s \geq k; \end{cases} \\ r_1 \geq r_2. \end{array} \right\}.$$

Put, for any real positive number $p \geq 1$,

$$\Omega_{k;p}^{(\varepsilon)} \equiv \Omega_k^{(\varepsilon)} \cap \{r_1, r_2 \geq p\}.$$

Obviously, $\Omega_{k;1}^{(\varepsilon)} = \Omega_k^{(\varepsilon)}$. Put $\Omega_{k;p}^{(0)} = \Omega_{k;p}$. The class $X_{(m-2/l_1)}$ of Banach spaces $L_{(r_1, r_2, l)}(Q_T)$ is defined by the equality

$$X_{(m-2/l_1)} \equiv \{L_{(r_1, r_2, l_1)}(Q_T); (r_1, r_2) \in \Omega_{(m-2/l_1)}, l_1 > 1\}.$$

We define the class $\widehat{X}_{(2-2/l_1);p}$ by the equality

$$\widehat{X}_{(2-2/l_1);p} = \bigcup_{(r_1, r_2) \in \Omega_{(2-2/l_1);p}} L_{(r_1, r_2, l_1)}(Q_T).$$

Let

$$R_{ij} = R_{ij}(r_{ij1}, r_{ij2}, l_1), \quad P_{ij} = P_{ij}(p_{ij1}, p_{ij2}, l), \quad i = 1, 2, \dots, n; \quad j = 1, 2,$$

where $p_{ijk} = 2r_{ijk}$, $k = 1, 2$; $l = 2l_1$,

$$R'_{ij} = R'_{ij}(r'_{ij1}, r'_{ij2}, l'_1), \quad P'_{ij} = P'_{ij}(p'_{ij1}, p'_{ij2}, l'),$$

$$R_3 = R_3(r_{31}, r_{32}, l_1), \quad R_4 = R_4(r_{41}, r_{42}, l'),$$

$$P_3 = P_3(p_{31}, p_{32}, l), \quad P_4 = P_4(p_{41}, p_{42}, l), \quad p_{kj} = 2r_{kj}.$$

Theorem 1. If $u(x, t) \in \dot{V}_2(Q_T)$ and the numbers q_1, q_2, l satisfy the conditions

$$1/l + s/2q_1 + (n-s)/2q_2 = n/4, \quad 2 \leq q_1, q_2, l,$$

then

$$\|u\|_{L(q_1, q_2, l)(Q_T)} \leq c \operatorname{vrai} \max_{0 \leq t \leq T} \|u\|_{L_2(D)}^{1-2/l} \|u_x\|_{L_2(Q_T)}^{2/l}.$$

Corollary. Under the condition of Theorem 1, the estimate

$$\|u\|_{L(q_1, q_2, l)(Q_T)} \leq c |u|_{Q_T}$$

holds.

Lemma 1. If the numbers q_1, q_2, l satisfy the conditions

$$1/l + s/2q_1 + (n-s)/2q_2 > n/4, \quad (3)$$

then one can choose numbers $\tilde{q}_1, \tilde{q}_2, \tilde{l}$ satisfying the conditions

$$1/\tilde{l} + s/2\tilde{q}_1 + (n-s)/2\tilde{q}_2 = n/4, \quad \tilde{l}/\tilde{q}_2 = l/q_2,$$

$$1/q_1 - 1/q_2 = 1/\tilde{q}_1 - 1/\tilde{q}_2. \quad (4)$$

Lemma 2. If the numbers q_1, q_2, l and $\tilde{q}_1, \tilde{q}_2, \tilde{l}$ satisfy conditions (3), (4) and $f \in L_{(\tilde{q}_1, \tilde{q}_2, \tilde{l})}(Q_T)$, then the estimate

$$\|f\|_{L(q_1, q_2, l)(Q_T)} \leq \|f\|_{L(\tilde{q}_1, \tilde{q}_2, \tilde{l})(Q_T)} \left\{ \int_0^T [\operatorname{mes} A(t)]^{\tilde{l}/q} dt \right\}^{(\tilde{l}-l)/\tilde{l}};$$

$A(t)$ is the set of points $x \in D$ where $|f(x, t)| > 0$.

Consider equation (1). Suppose the coefficients satisfy the conditions

$$\nu \xi_i \xi_i \leq a_{ij} \xi_i \xi_j \leq \mu \xi_i \xi_i, \quad \nu, \mu = \text{const} > 0, \quad a_{ij} = a_{ji}; \quad (5)$$

$$\|a_i^2\|_{L_{R_{i1}}(Q_T)}, \quad \|b_i^2\|_{L_{R_{i2}}(Q_T)}, \quad \|a\|_{L_{R_3}(Q_T)} \leq \mu_1; \quad (6)$$

$$\left\| \left(\sum f_i^2 \right)^{1/2} \right\|_{L_2(Q_T)}, \quad \|f\|_{L_{R_4}(Q_T)} \leq \mu_1, \quad (7)$$

where

$$L_{R_3}(Q_T), L_{R_{ij}}(Q_T) \in X_{(2-2/l_1)}, \quad i = 1, 2, \dots, n; \quad j = 1, 2; \quad (8)$$

$$L_{R_4} \in X_{((n+4)/2-2/l')}. \quad (9)$$

Denote

$$L_1(u, \eta) \equiv \int_D [(a_{ij} u_{x_j} + a_i u) \eta_{x_i} + (b_i u_{x_i} + a u) \eta] dx,$$

$$L_2(\hat{f}, \eta) \equiv \int_D (f \eta + f_i \eta_{x_i}) dx.$$

Theorem 2. Suppose $u(x, t) \in \dot{V}_2(Q_T)$, and u satisfies, for almost all t_1 and t_2 in $[0, T]$, including for $t_1 = 0$, the inequalities

$$-\frac{1}{2} \int_D u^2(x, t) dx \Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} [L_1(u, u) + L_2(\hat{f}, u)] dt \leq 0. \quad (10)$$

The coefficients a_{ij}, b_i, a_i, a and the free terms f and f_i satisfy conditions (5)–(9). Then

$$|u|_{Q_T} \leq C[\|u(x, 0)\|_{L_2(D)} + \|\bar{f}\|_{L_2(Q_T)} + \|f\|_{L_{R_4}(Q_T)}], \quad (11)$$

where C is a constant depending on $n, \nu, \mu, \mu_1, R_{ij}, R_3, R_4$.

Theorem 3. For any generalized solution $u(x, t)$ from $\dot{V}_2^{1,0}$ of problem (1)–(2), the inequality

$$|u|_{Q_T} \leq C[\|\psi_0\|_{L_2(D)} + \|\bar{f}\|_{L_2(Q_T)} + \|f\|_{L_{R_4}(Q_T)}]$$

holds, provided that assumptions (5)–(9) are fulfilled with respect to equation (1).

Theorem 4. If conditions (5)–(9) are fulfilled and $\psi_0 \in L_2(D)$, then problem (1), (2) has a solution from $V_2(Q_T)$.

Theorem 5. Under conditions (5)–(9), any generalized solution $u(x, t)$ of problem (1), (2) from $\dot{V}_2(Q_T)$ belongs to $\dot{V}_2^{1,1/2}(Q_T)$, and problem (1), (2) is uniquely solvable in $\dot{V}_2^{1,1/2}(Q_T)$, if $\psi_0(x) \in L_2(D)$.

Theorem 6. If the coefficients of the equation satisfy conditions (5)–(9), then the boundary-value problem for (1), (2) cannot have two distinct generalized solutions from $V_2(Q_T)$.

Theorem 7. Suppose $u(x, t)$ is a generalized solution from $\dot{V}_2(Q_T)$ of problem (1), (2), and suppose a_{ij} , a_i , and f_i satisfy conditions (5)–(9), while $b_i(x, t)$, $a(x, t)$, and $f(x, t)$ satisfy the conditions

$$\left\| \sum b_i^2, a \right\|_{L_{(r_1, r_2, l_1)}^*(Q_T)} \leq \mu_1, \quad l_1 \geq r_2 \geq r_1; \quad \|f\|_{L_{(r_{41}, r_{42}, l')}^*(Q_T)} \leq \mu_1;$$

$$L_{(r_{11}, r_2, l_1)}(Q_T) \in X_{(2-2/l_1)}; \quad L_{(r_{41}, r_{42}, l')}(Q_T) \in X_{((n+2)/2-1/l_1)},$$

where

$$L_{(m_1, m_2, \rho)}^*(Q_T) = L_{(\rho, m_2, m_1)}(0 \leq t \leq T, \tilde{D}_2, \tilde{D}_1).$$

Then the function $u(x, t)$ is an element of $\dot{W}_2^{1,-1/2}(Q_T)$.

Theorem 8. Suppose that for all operators

$$\mathcal{L}^m u = u_t - \frac{\partial}{\partial x_i} (a_{ij}^m u_{x_j} + a_i^m u) + b_i^m u_{x_i} + a^m u, \quad m = 1, 2, \dots,$$

the conditions of Theorem 4 are satisfied with the same constants. Suppose that $a_{ij}^m(x, t)$, remaining uniformly bounded, converge almost everywhere to a_{ij} , and that the functions $a_i^m, b_i^m, a^m, f_i^m, \psi_0^m$ converge to a_i, b_i, a, f, ψ_0 in the norms of the spaces to which they belong under the conditions of Theorem 4. Then the generalized solutions u^m from $V_2^{1,0}(Q_T)$ of the problems

$$\mathcal{L}^m u \equiv \partial f_i^m / \partial x_i - f^m, \quad u|_{S_T} = 0, \quad u|_{t=0} = \psi_0^m$$

converge strongly in $V_2^{1,0}(Q_T)$ to the generalized solution $u(x, t)$ of the limiting problem (1)–(2).

Theorem 9. Suppose that the coefficients of equation (1) satisfy the conditions

$$\nu \xi_i \xi_i \leq a_{ij} \xi_i \xi_j \leq \mu \xi_i \xi_i, \quad \nu, \mu = \text{const} > 0, \quad a_{ij} = a_{ji},$$

$$\|a, f, a_i^2, f_i^2\|_{L(r_1, r_2, l)(Q_T)} < \mu_1, \quad \|b_i^2\|_{L(\bar{r}_1, \bar{r}_2, \bar{l})(Q_T)} \leq \mu_1,$$

where $(r_1, r_2) \in \Omega_{(2-2/l_1)}^{(\varepsilon)}$, $(\bar{r}_1, \bar{r}_2) \in \Omega_{(2-2/\bar{l}_1)}$.

Then, for every generalized solution $u(x, t)$ from $V_2^{1,0}(Q_T)$ of equation (1), not exceeding k_0 on Γ_T , $\text{vrai max } u(x, t)$ is finite and is estimated from above by a constant determined only by $n, k_0, \nu, \mu, \mu_1, r_1, r_2, l_1, \bar{r}_1, \bar{r}_2, \bar{l}_1$.

Theorem 10. Suppose that $u(x, t)$ is a generalized solution from $V_2^{1,0}(Q_T)$ of equation (1), whose coefficients a_{ij}, b_i, a satisfy the conditions

$$\nu \xi_i \xi_i \leq a_{ij} \xi_i \xi_j \leq \mu \xi_i \xi_i, \quad \nu, \mu = \text{const} > 0,$$

$$\|b_i^2, a\|_{L(r_1, r_2, l)(Q_T)} \leq \mu_1, \quad (r_1, r_2) \in \Omega_{(2-2/l_1)},$$

$$a(x, t) \geq 0, \quad a_i = f_i = f = 0.$$

Then for almost all (x, t) in Q_T

$$\min \left\{ 0, \text{vrai min}_{\Gamma_T} u(x, t) \right\} \leq u(x, t) \leq \max \left\{ 0, \text{vrai max}_{\Gamma_T} u(x, t) \right\}.$$

Theorem 11. Suppose that $u(x, t)$ is a generalized solution from $V_2^{1,0}(Q_T)$ of equation (1), whose coefficients and free terms satisfy the conditions of Theorem 9. Then, for any cylinder Q' lying at a positive distance d from Γ_T , the quantity $\text{vrai max}_{Q'} |u|$ is estimated from above by a constant depending only on $n, \nu, \mu, \mu_1, r_1, r_2, l_1, \bar{r}_1, \bar{r}_2, \bar{l}_1, d, \|u\|_{L_2(Q_T)}$.

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