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1968

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Abstract

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UDC 513.83

MATHEMATICS

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ON DYADIC BICOMPACTS

(Presented by Academician P. S. Aleksandrov on 15 II 1968)

In this paper a new approach is given to the proof of the well-known theorem of A. S. Esenin-Vol'pin ⁽¹⁻³⁾ on the metrizable of a dyadic bicomact with the first axiom of countability, and new results are obtained. By $\omega(X)$ we denote the weight of the space X , by $|X|$ its cardinality, by $s(X)$ its density, i.e. the least of the cardinalities of everywhere dense subsets in X ; $\chi(x, X)$ is the character of the space X at the point $x \in X$.

Proposition 1. *Let X be a continuous image of D^τ , and let $A \subset X$ be arbitrary. Then there exists a $Z \subset X$ such that: a) Z is a continuous image of D^t ; b) $A \subseteq Z$; c) $s(A) \geq s(Z)$.*

Proof. Let $f : D^\tau \rightarrow X$ be a continuous mapping; let $A^* \subset A$ be a subset everywhere dense in A , for which $|A^*| = s(A)$, and let $L \subset D^\tau$ be a subspace mapped by f one-to-one onto A^* (obviously such a subspace exists). Recall that D^τ is a topological group, and take its smallest closed subgroup G_L containing L . It is clear that $s(G_L) \leq |L| = |A^*| = s(A)$. Consequently, $s(f(G_L)) \leq s(A)$. It is well known that the space G_L is homeomorphic to $D^{\tau'}$ for some $\tau' \leq \tau$ (see the direct proof in E. Hewitt's paper ⁽⁶⁾). Hence one may take $f(G_L)$ for Z .

Remark. Proposition 1 remains valid if X is assumed to be the image of some topological group and, instead of a), it is required that Z be the image of some topological group.

Lemma 1. *Let X be a topological space; P a class of its subspaces, $P \ni X$, and for every $A \subset X$ there exists an $\tilde{A} \in P$, $\tilde{A} \supset A$, whose density $s(\tilde{A})$ does not exceed the density $s(A)$ of the space A . If $s(X) > \tau_1$, then there exists $Y_1 \in P$ for which $s(Y_1)$ is a regular cardinal number greater than τ_1 .*

Proof. Put

$$P(\tau_1) = \{Y \in P : \text{if } Y' \subseteq X \text{ and } Y' \supseteq Y, \text{ then } s(Y') \geq s(Y) > \tau_1\},$$

and let $Y_1 \in P(\tau_1)$ be chosen so that $s(Y_1) \leq s(Y)$ for every $Y \in P(\tau_1)$. Then $s(Y_1)$ is a regular cardinal number.

Suppose the contrary. Choose A_1 everywhere dense in Y_1 so that $|A_1| = s(Y_1)$, and let

$$A_1 = \bigcup \{A_1^\alpha : \alpha \in M\},$$

where $|M| < |A_1|$ and $|A_1^\alpha| < |A_1|$. For every A_1^α , $\alpha \in M$, find $\tilde{A}_1^\alpha \in P$ of least density satisfying the condition of the lemma:

$$s(\tilde{A}_1^\alpha) \leq |A_1^\alpha|, \quad \tilde{A}_1^\alpha \supset A_1^\alpha.$$

Then from $s(\tilde{A}_1^\alpha) < |A_1| = s(Y_1)$ and $\tilde{A}_1^\alpha \in P$ it follows that $\tilde{A}_1^\alpha \notin P(\tau_1)$, whence $s(\tilde{A}_1^\alpha) \leq \tau_1$ by the choice of $\tilde{A}_1^\alpha$. Since the last relation holds for all $\alpha \in M$, we have

$$s\left(\bigcup \{\tilde{A}_1^\alpha : \alpha \in M\}\right) \leq \max(\tau_1 |M|) < s(Y_1).$$

But

$$\bigcup \{\tilde{A}_1^\alpha : \alpha \in M\} \supseteq Y_1$$

and $Y_1 \in P(\tau_1)$ —a contradiction.* $P(\tau_1) \neq \Lambda$, for $X \in P(\tau_1)$.

We shall say that a topological space **satisfies Shanin's condition** ⁽⁵⁾ if every family, completely ordered by inclusion (decreasing), of its nonempty open sets whose intersection is empty contains a countable cofinal part.

Lemma 2. *If a topological space X satisfies Shanin's condition and its density $s(X)$ is regular and is not equal to $\aleph_0$, then*

$$s(X) \leq \partial(x, X)$$

for some point $x \in X^{**}$.

* From Proposition 1 and Lemma 1 it follows that the density of every (dyadic) bicomact X is the least upper bound of those regular cardinal numbers which are densities of (dyadic) bicomacts contained in X . An analogous assertion is true for arbitrary spaces.

** By $\partial(x, X)$ is denoted the least cardinal number τ such that, if $x \in [M]$, where $M \subset X$, then there exists $M' \subset M$ with the properties $|M'| \leq \tau$ and $x \in [M']$.

Proof. Let $A \subset X$, let A be everywhere dense in X , and let $|A| = s(X) = \tau$. We shall assume that the set A is well ordered according to the minimal ordinal type of the cardinal number τ (by the relation $<$). For each $y \in A$ put

$$A(y) = \{y' \in A : y' < y\}$$

and

$$B(y) = [A(y)].$$

Then, if $A_0 \subset A$ and $|A_0| < \tau$, there exists $y_0 \in A$ for which $A(y_0) \supset A_0$ (by the regularity of the cardinal τ). Suppose now that $\partial(x_0, X) < s(X) = \tau$ for

some $x_0 \in X$. From what was said above (and from the definition of $\partial(x_0, X)$) it follows that there exists $y'_0 \in A$ for which $[A(y'_0)] \ni x_0$, i.e. $B(y'_0) \ni x_0$. If we assume that the relation $\partial(x, X) < \tau$ holds for all $x \in X$, then we obtain

$$\bigcup \{B(y) : y \in A\} = X.$$

But from $|A(y)| < |A| = s(X)$ (true for all $y \in A$) it follows that $B(y) \neq X$. Hence $\{G(y) : y \in A\}$, where $G(y) = X \setminus B(y)$, is a decreasing (since $B(y) \subseteq B(y')$ for $y < y'$) sequence of nonempty open subsets of X , not cofinal with the countable one, with empty intersection.

Definition 1. By $t(X)$, where X is a topological space, we shall denote the least cardinal number τ such that, whatever the set $A \subset X$ and the point $x \in [A]$, there exists $A' \subset A$ for which $|A'| \leq \tau$ and $x \in [A']$. We shall call $t(X)$ the **tightness** of the space X . It is obvious that, if $Y \subset X$, then $t(Y) \leq t(X)$.

Theorem 1. Let X be a space satisfying the following condition:

(III). If $A \subset X$, then there exists a $Y \subset X$, $Y \supset A$, satisfying Shanin's condition, for which $s(A) \geq s(Y)$.

Then

$$t(X) \geq s(X).$$

Proof. By contradiction: suppose $s(X) > t(X)$. Denote by P the collection of all subspaces of the space X satisfying Shanin's condition, and put $\tau_1 = t(X)$. On the basis of Lemma 1 we conclude that there exists $Y_1 \subset X$, $Y_1 \in P$, for which $s(Y_1) > \tau_1 = t(X)$, and $s(Y_1)$ is a regular cardinal number. Then

$$s(Y_1) > t(Y_1),$$

which contradicts Lemma 2. The theorem is proved.*

Definition 2. A space X is called a π -space if two conditions are satisfied for it: a) for each $A \subset X$ there exists a subspace $\bar{A} \subset X$ containing it, satisfying Shanin's condition and having density not exceeding the density of A ; b) the π -weight of the space X is equal to the weight of X .

As is known, all dyadic bicomponents satisfy conditions a) and b), and consequently all of them are π -spaces.

It is known that the π -weight** of a space X (unlike weight, generally speaking) is very simply connected with the density of X and the upper bound of the characters of points of X —namely, the π -weight of X does not exceed the maximum of the following two numbers.

Theorem 2. The weight of a π -space X is equal to

$$\tau = \sup\{\chi(x, X) : x \in X\}.$$

Proof. We have

$$\sup\{\chi(x, X) : x \in X\} \geq t(X).$$

By Theorem 1,

$$t(X) \geq s(X).$$

Thus, $\tau \geq s(X)$. But

$$\omega(X) = \pi(X) \leq \max(\tau, s(X)).$$

Consequently, $\omega(X) \leq \tau$. Since always

$$\omega(X) \geq \sup\{\chi(x, X) : x \in X\},$$

we conclude that

$$\omega(X) = \tau.$$

Corollary 1 (A. S. Esenin-Vol'pin). *The weight of a dyadic bicomact does not exceed the upper bound of the characters of its points.*

* We note that we have proved the general assertion: if a cardinal-valued function δ is defined on the class of all bicomacts, is monotone with respect to closed subspaces, and $\delta(X) \geq s(X)$ for any (dyadic) bicomact X whose density $s(X)$ is regular, then $\delta(X) \geq s(X)$ for any (dyadic) bicomact X (such is the tightness $t(X)$).

** The π -weight of a space X is the least cardinal number equal to the cardinality of a system of open subsets of X that is dense in X . A system σ of open subsets of X is dense in X if for any open set $\Gamma \subseteq X$ there exists $U \in \sigma$ for which $U \subseteq \Gamma$ (see (7)).

Corollary 2. If a topological space X and each of its closed subspaces satisfy Shanin' s condition, then

$$s(X) \leq \sup\{\partial(x, X) : x \in X\}.$$

Lemma 3. If a space X is separable, then it also satisfies Shanin' s condition.

From Lemma 3 and Corollary 2 there further follows

Theorem 3. In order that a Fréchet-Urysohn space and each of its closed subspaces satisfy Shanin' s condition, it is necessary and sufficient that it be hereditarily separable with respect to closed subsets.

Theorem 4. In order that a space X and each of its subspaces satisfy Shanin' s condition, it is necessary and sufficient that X be hereditarily separable with respect to all subspaces.

Proof. a) Let X and each of its subsets satisfy Shanin' s condition. Suppose that there is a nonseparable subspace in X . Then, by Lemma 1, there exists a subspace $\bar{A} \subset X$ whose density $s(\bar{A})$ is equal to an uncountable regular cardinal number; but then there exists a set $B \subseteq \bar{A}$ that does not satisfy Shanin' s condition. The contradiction obtained proves that X is hereditarily separable.

b) The converse assertion follows immediately from Lemma 3.

Remark 1. It is not hard to verify that a space X satisfies Shanin's condition if and only if every system σ of regular cardinality, consisting of sets open in X , contains a subsystem of the same cardinality with nonempty intersection.

Remark 2. It is easy to see that Suslin's condition is formally weaker than Shanin's condition. Therefore it is natural to pose the following problem:

Is every bicomcompact space with the first axiom of countability, satisfying Suslin's condition hereditarily with respect to closed subsets, separable?

This question, as we shall now show, is connected in one direction with the well-known Suslin problem. Indeed, a nonseparable Suslin continuum, if it exists, satisfies Suslin's condition hereditarily with respect to closed subsets, satisfies the first axiom of countability, and its weight and π -weight are equal*. The Suslin continuum itself even satisfies Shanin's condition. However, we do not know whether or not our problem is equivalent to the Suslin problem.

Definition 3. A point $x \in X$ will be called a **point of extremal accumulation** for a set $M \subset X$ if

$$\inf\{Ox \cap M : Ox \text{ is a neighborhood of } x\} > \inf\{Oy \cap M : Oy \text{ is a neighborhood of } y\}$$

for every $y \neq x, y \in X$.

Definition 4. We shall say that X is a χ -space if from $x \in X, M \subset X, x \in [M]$ it follows that there exists $M' \subset M$ for which x is the unique point of extremal accumulation in X .

An example of a non- χ -space is βN . Every Fréchet-Urysohn space is a χ -space**. It is clear that every closed subspace of a χ -space is itself a χ -space. Therefore not every dyadic bicomcompactum is a χ -space (some of them contain βN).

Of fundamental importance is the following obvious assertion.

Proposition 2. If X is a χ -space, then $|X| \leq 2^{s(X)}$.

In proving the following assertion we assume that

$$2^{\tau_1} > 2^{\tau_2},$$

if $\tau_1 > \tau_2$ (which is weaker than the generalized continuum hypothesis).

* Cf. these conditions with the references of the direct Theorem 3.

** The space of all transfinite ordinals not exceeding the first uncountable is a χ -space, but not a Fréchet-Urysohn space.

Theorem 5. If a bicomcompactum X is a χ -space and for every $A \subset X$ there exists $\tilde{A} \supset A, s(\tilde{A}) \geq |A|$, satisfying Shanin's condition, then

$$[\{x \in X : \chi(x, X) \leq t(X)\}] = X.$$

Proof. It has already been proved that $t(X) \geq s(X)$. Moreover, $\omega(X) \geq t(X)$. If

$$X \neq [\{x \in X : \chi(x, X) \leq t(X)\}],$$

then, by the Čech-Pospíšil theorem ^(8,9),

$$|X| \geq 2^{t(X)+1}.$$

But by Proposition 2,

$$|X| \leq 2^{s(X)},$$

i.e.

$$2^{t(X)+1} \leq 2^{s(X)}.$$

But $t(X) + 1 > s(X)$ —a contradiction.

Remark. The advantage of the formulations and arguments given above is that the topological background of the equality $t(X) = \omega(X)$ is touched upon rather deeply. However, if one restricts oneself to dyadic bicompacta, the final result can be obtained without using the continuum hypothesis at all.

Definition 5. The **weak tightness** $t_0(X)$ of a topological space X is the least of the cardinal numbers τ for which:

(*) if $M \subset X$ and $M \neq [M]$, then there exist $M' \subset M$ and $x' \in X$ such that

$$|M'| \leq \tau \quad \text{and} \quad x' \in [M'] \setminus M.$$

It is clear that always

$$t_0(X) \leq t(X) \leq \sup\{\chi(x, X) : x \in X\} \leq \omega(X).$$

Weak tightness is sometimes more convenient than tightness, for the following is true.

Proposition 3. *If Y is a quotient space of a space X , then*

$$t_0(Y) \leq t_0(X).$$

Now the first of the following two theorems follows from the second.

Theorem 6. *If a dyadic bicompactum Y is a quotient space of a space X , then*

$$\omega(Y) \leq \sup\{\chi(x, X) : x \in X\}.$$

Theorem 7. *For every dyadic bicompactum X ,*

$$t_0(X) = t(X) = \sup\{\chi(x, X) : x \in X\} = \omega(X).$$

Corollary. *If for a dyadic bicompactum X*

$$t_0(X) = \aleph_0,$$

then also

$$\omega(X) \leq \aleph_0.$$

This contains, in particular, the result of B. A. Efimov: a dyadic bicom pactum which is a Fréchet–Urysohn space is metrizable.

Proposition 4*. $t_0(X) \leq \tau$ if and only if X is a quotient space of a space $\widetilde{X}$ of local cardinality $\leq \tau$ (briefly: $|\widetilde{X}|_{\text{loc}} \leq \tau$).

This characterization is useful: since the property of having local cardinality $\leq \tau$ is inherited by all subspaces, and a quotient mapping on the full preimage of an open set induces a quotient mapping, we derive from Proposition 4:

Proposition 5. If U is open in X , then

$$t_0(U) \leq t_0(X).$$

Theorem 6 can be given the following form:

Theorem 6'. If a dyadic bicom pactum Y is a quotient space of a space X whose local cardinality does not exceed τ , then also

$$\omega(Y) \leq \tau.$$

Question 1. Does there exist a bicom pactum X which is simultaneously a χ -space and a π -space, for which

$$t(X) \neq \omega(X)?$$

Question 2. Does there exist a nonmetrizable dyadic bicom pactum which is a χ -space?

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Received
15 I 1968

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- * This proposition underlies the proof of Proposition 3.

Note: Figure translations are in progress. See original paper for figures.

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