

THREE THEOREMS ON NONLINEAR EQUATIONS

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Abstract

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MATHEMATICS

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THREE THEOREMS ON NONLINEAR EQUATIONS

WITH MONOTONE OPERATORS

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The paper contains three theorems. The first theorem establishes the convergence of the method of least squares for equations with monotone operators and represents a further development of the results of S. G. Mikhlin and A. Langenbach.

The second theorem gives conditions for the solvability of equations with monotone operators (cf. ⁽⁵⁻⁸⁾). In contrast to existing works, reflexivity of the space is not assumed in it. Moreover, monotonicity is understood in a more general sense. The third theorem gives sufficient conditions for uniform convexity of a space in terms of monotone operators.

1. Let E be a real reflexive separable Banach space; let E^* be its conjugate; suppose that linear combinations of the elements $x_1, \dots, x_n, \dots$ form an everywhere dense set $N \subseteq E$, and that the elements x_i are linearly independent in any finite number. Let $N \subseteq M \subseteq E$, and let the operator F act from the set M into E^* . As is known ⁽¹⁾, the method of least squares for solving the equation $F(x) = \theta$ consists in seeking an absolute minimum of the function

$$\left\| F \left(\sum_{i=1}^n a_i x_i \right) \right\|$$

of n real variables $a_1, \dots, a_n$, for fixed $x_1, \dots, x_n$. If the minimum is attained at the element

$$u_n^0 = \sum_{i=1}^n a_i^0 x_i$$

(not necessarily unique), then u_n^0 is an approximate solution obtained by the method of least squares. The sequence $\{x_i\}$ is called F -complete ⁽¹⁾ in E if, for every $\varepsilon > 0$, there exist a natural number n and real numbers $\alpha_1, \dots, \alpha_n$ such that

$$\left\| F \left(\sum_{i=1}^n \alpha_i x_i \right) \right\| < \varepsilon.$$

Theorem 1. Let the operator F satisfy the condition of strong monotonicity

$$(x - y, F(x) - F(y)) \geq \gamma(\|x - y\|)\|x - y\|, \quad \forall x, y \in M,$$

where $\gamma(t)$ is a continuous strictly increasing function for $t \geq 0$, $\gamma(0) = 0$,

$$\lim_{t \rightarrow +\infty} \gamma(t) = +\infty.$$

Suppose that at least one of the conditions I, II holds:

I. The restriction of the operator F to every finite-dimensional subspace of M is a continuous operator; there exists $x^* \in M$, $F(x^*) = \theta$; the sequence $\{x_i\}$ is F -complete.

II. $M = E$ and F is a continuous operator (one can prove that in this case there exists a unique solution x^* of the equation $F(x) = \theta$).

Then the approximate solutions u_n^0 , obtained by the method of least squares, exist and

$$\lim_{n \rightarrow \infty} \|u_n^0 - x^*\| = 0.$$

Proof. If condition I is fulfilled, then by virtue of F -completeness there exists a sequence of elements

$$u_n = \sum_{i=1}^n a_i x_i$$

such that $F(u_n) \rightarrow \theta$.

If condition II is fulfilled, then, since N is everywhere dense in E , there also exists a sequence of elements $u_n = \sum_{i=1}^n a_i x_i$ such that $u_n \rightarrow x^*$, and therefore $F(u_n) \rightarrow F(x^*) = \theta$. Thus, in any case $F(u_n) \rightarrow \theta$. Since $\|F(u_n^0)\| \leq \|F(u_n)\|$, it follows that $F(u_n^0) \rightarrow \theta$. From the condition of strong monotonicity it immediately follows that $\lim_{n \rightarrow \infty} \|u_n^0 - x^*\| = 0$.

2. Let X be a Banach space, and let Z be a real Banach space with cone K , i.e. K is a closed convex set such that, for $x \in K$, $\forall \lambda \geq 0$ we have $\lambda x \in K$; for $x \neq \theta_Z$ we have $(-x) \notin K$ (see [4], p. 13). Suppose that in X

there is defined an operation of multiplication $z = xy$, $\forall x, y \in X$, linear in each variable, with values of the product in Z . Let multiplication and the norms possess the properties $\|xy\|_Z \leq \|x\|_X \|y\|_X$, $\|x^2\|_Z = \|x\|_X^2$, $x^2 \geq \theta_Z$, $\forall x, y \in X$, and let the norm in Z be monotone, i.e. if $w \geq z \geq \theta_Z$, then $\|w\|_Z \geq \|z\|_Z$, $\forall z, w \in Z$. A pair of spaces X, Z satisfying the requirements listed above will be called an admissible pair. For example, if X is Hilbert, $Z = R^1$, $K = \{z : z \geq 0, z \in R^1\}$, and xy is the scalar product, then X, Z is an admissible pair of spaces. Two spaces of continuous functions $X \equiv Z \equiv C[a, b]$ with the cone of nonnegative functions and the usual multiplication give another example of an admissible pair of spaces.

Theorem 2. Let X, Z be an admissible pair of spaces, and let the operator $F(x)$, acting from X into X , in each ball $D_r\{x : \|x\| \leq r, x \in X\}$ satisfy the Lipschitz and strong monotonicity conditions:

$$[F(x) - F(y)]^2 \leq m^2(r)(x - y)^2, \quad \forall x, y \in D_r; \quad (1)$$

$$[F(x) - F(y)](x - y) - k(r)(x - y)^2 \geq \theta_Z, \quad \forall x, y \in D_r; \quad (2)$$

$$(x - y)[F(x) - F(y)] - k(r)(x - y)^2 \geq \theta_Z, \quad \forall x, y \in D_r, \quad (3)$$

where $k(r) > 0$, $m(r) > 0$, $\forall r > 0$, and $\lim_{r \rightarrow +\infty} rk(r) = +\infty$.

Then the operator $F(x)$ realizes a homeomorphic mapping of the space X onto X .

Proof. From inequality (1) it follows that $F(x)$ satisfies the usual Lipschitz condition $\|F(x) - F(y)\|_X \leq m(r)\|x - y\|_X$, $\forall x, y \in D_r$, and from inequality (2) we have $k(r)\|x - y\|_X^2 \leq \|F(x) - F(y)\|_X \|x - y\|_X$, $\forall x, y \in D_r$.

Therefore $k(r) \leq m(r)$, $\forall r > 0$. Since $\lim_{r \rightarrow +\infty} rk(r) = +\infty$, there is $r_0 > 0$ such that the inequality holds:

$$\|F(\theta)\|_X \left[1 + \sqrt{1 - k^2(r_0)/m^2(r_0)} \right] < r_0 k(r_0). \quad (4)$$

Put

$$\Phi(x) = x - \frac{k(r_0)}{m^2(r_0)} F(x) \quad (5)$$

and consider the sequence

$$x_n = \Phi(x_{n-1}), \quad x_0 = \theta. \quad (6)$$

By a simple calculation (using inequalities (1), (2), (3)) we obtain

$$[\Phi(x) - \Phi(y)]^2 \leq (x - y)^2 [1 - k^2(r_0)/m^2(r_0)], \quad \forall x, y \in D_{r_0}.$$

Putting $\mu_0(r_0) = \sqrt{1 - k^2(r_0)/m^2(r_0)}$, from the last inequality we obtain

$$\|\Phi(x) - \Phi(y)\|_X \leq \mu_0(r_0) \|x - y\|_X, \quad \forall x, y \in D_{r_0}, \quad (7)$$

where $\mu_0(r_0) < 1$.

From inequalities (4), (5), (6) we have:

$$\|x_1\|_X = \|\Phi(\theta)\|_X \leq \frac{k(r_0)}{m^2(r_0)} \frac{r_0 k(r_0)}{1 + \sqrt{1 - k^2(r_0)/m^2(r_0)}} = [1 - \mu_0(r_0)]r_0, \quad (8)$$

i.e. $x_1 \in D_{r_0}$. Assuming that $x_1, \dots, x_{n-1} \in D_{r_0}$, we find

$$\begin{aligned} \|x_n\|_X &\leq \|x_n - x_{n-1}\|_X + \|x_{n-1} - x_{n-2}\|_X + \dots + \|x_1 - x_0\|_X \leq \\ &\leq \sum_{i=1}^n \mu_0^{i-1}(r_0) \|x_1 - x_0\|_X. \end{aligned}$$

Hence, and from the estimate for x_1 , we have

$$\|x_n\|_X \leq \sum_{i=1}^n \mu_0^{i-1}(r_0) [1 - \mu_0(r_0)]r_0 = [1 - \mu_0^n(r_0)]r_0 < r_0,$$

so that, by induction, all $x_n \in D_{r_0}$. Using inequality (7), in the standard way one can prove that x_n is a fundamental sequence, and therefore there exists $\lim_{n \rightarrow \infty} x_n = x^*$. Taking into account the continuity of $\Phi(x)$, we obtain from this:

$$x^* = \Phi(x^*) = x^* - \frac{k(r_0)}{m^2(r_0)} F(x^*),$$

or $F(x^*) = \theta$. If φ is an arbitrary element of X , then the operator $F_1(x) = F(x) - \varphi$, evidently, satisfies all the conditions of the theorem, and therefore the equation $F_1(x) = \theta$ is solvable in X . Thus, the range of the operator $F(x)$ is the whole space X . The fact that $F(x)$ realizes a homeomorphic mapping is obtained quite simply if one uses inequality (2). The theorem is proved.

3. Let E be a real Banach space; E^* , its conjugate; and let the operator $F : E \rightarrow E^*$ be strongly monotone in the following sense:

$$(x - y, F(x) - F(y)) \geq \gamma(\|x - y\|)\|x - y\|, \quad \forall x, y \in E. \quad (9)$$

(The notation (x, z) means the value of the linear functional $z \in E^*$ at the element $x \in E$.) Such operators have been studied in a number of works. The question naturally arises whether such an operator exists in every space. A partial answer to the question is given by

Theorem 3. *Suppose there exists an operator F defined on the whole space E and satisfying inequality (9). Suppose, moreover, that the following conditions are fulfilled: a) F is the gradient of a functional $f : E \rightarrow R^1$; b) the functional $f(x)$ is uniformly continuous on every bounded set of the space E ; c) for any fixed $x, h \in E$ the function $(F(x + th), h)$ is continuous in t ; d) the real function γ of a nonnegative argument is such that the function*

$$\nu(R) = \int_0^1 \gamma(tR) dt$$

is continuous, strictly increasing for $R \geq 0$, $\nu(0) = 0$, and $\lim_{R \rightarrow \infty} \nu(R) = +\infty$.

Then in the space E one can introduce a norm equivalent to the original one, so that in the new norm the space E will be uniformly convex.

Proof. Carrying out simple transformations, and using inequality (9) (cf. (2)), we obtain

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{1}{2}(x + y)\right) \geq \frac{1}{4}\nu(\|x - y\|)\|x - y\|, \quad \forall x, y \in E. \quad (10)$$

Without loss of generality, one may assume that the functional f is even and $f(\theta) = 0$; otherwise one may introduce a new functional $\varphi(x) = \frac{1}{2}[f(x) + f(-x)] - f(\theta)$, which also satisfies inequality (10). Setting $y = -x$ in (10), we find $f(x) \geq \frac{1}{2}\nu(2\|x\|)\|x\|$, whence it follows that

$$\lim_{\|x\| \rightarrow \infty} f(x) = +\infty. \quad (11)$$

It is not difficult to see that the functional f is nondecreasing along every ray issuing from zero. Consider now the set

$$K = \{x : f(x) \leq 1, x \in E\}. \quad (12)$$

By virtue of inequality (10), K is a convex set. Since the functional f is continuous and $f(\theta) = 0$, θ is an interior point of the set K . Consider the Minkowski functional $m(x)$ of the set K : $m(x) = \inf a_x$, $f(x/a_x) \leq$

≤ 1 , $a_x > 0$. It is known that $m(x) \geq 0$, $m(x) < \infty$, $m(\lambda x) = \lambda m(x)$, $m(x + y) \leq m(x) + m(y)$, $\forall x, y \in E$, $\forall \lambda \geq 0$. By virtue of the evenness of the functional f we have $m(\lambda x) = |\lambda| m(x)$ for all real λ . From the above-mentioned properties of the functional f it follows easily that $m(x) = 0$ if and only if $x = \theta$. Thus, $m(x)$ is a new norm in E . It is easy to verify that it is equivalent to the original norm $\|\cdot\|$. We shall now show that, in the norm $m(x)$, the space E is uniformly convex. Let

$$m(x_n) \leq 1, \quad m(y_n) \leq 1, \quad m(1/2(x_n + y_n)) \rightarrow 1. \quad (13)$$

From these relations it follows that

$$m(x_n) \rightarrow 1, \quad m(y_n) \rightarrow 1. \quad (14)$$

We shall now show that if $m(x_n) \rightarrow 1$, then also $f(x_n) \rightarrow 1$. Let $m(x_n) \rightarrow 1$. The inequality $f(x_n) > 1$ is impossible, for $1/m(x_n) \geq 1$, and therefore, by virtue of the nondecrease of the functional $f(x)$ along each ray issuing from θ , we have $f(x_n/m(x_n)) > 1$, which contradicts the definition of $m(x)$. Thus, $f(x_n) \leq 1$. Suppose that, for some subsequence $\{x_{n_k}\}$, the inequality $f(x_{n_k}) \leq \beta_0 < 1$, where $\beta_0 > 0$, holds. Since $\lim_{n_k \rightarrow \infty} m(x_{n_k}) = 1$, for the given $\delta_1 > 0$ there is an N such that for all $n_k \geq N$ we shall have

$$|1 - 1/m(x_{n_k})| < \delta_1. \quad (15)$$

The set K is bounded by virtue of relation (11). Therefore there exists an $M > 0$ such that $\|x_{n_k}\| \leq M$ for all n_k . By virtue of the uniform continuity of f on the set K , for the given $\varepsilon > 0$, $\varepsilon + \beta_0 < 1$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in K$, $\|x - y\| < \delta$. Taking $\delta_1 = \delta/M$ in (15), we obtain for all $n_k \geq N$

$$\|x_{n_k} - x_{n_k}/m(x_{n_k})\| < M\delta_1 = \delta.$$

Since $x_{n_k}, x_{n_k}/m(x_{n_k}) \in K$, by the uniform continuity of f on K we have, for all $n_k \geq N$,

$$|f(x_{n_k}) - f(x_{n_k}/m(x_{n_k}))| < \varepsilon,$$

whence

$$f(x_{n_k}/m(x_{n_k})) < f(x_{n_k}) + \varepsilon \leq \beta_0 + \varepsilon,$$

which contradicts the definition of $m(x_{n_k})$. Thus, $\lim_{n \rightarrow \infty} f(x_n) = 1$. Hence, from relations (13), (14), it follows that

$$f(x_n) \rightarrow 1, \quad f(y_n) \rightarrow 1, \quad f(1/2(x_n + y_n)) \rightarrow 1.$$

Putting $x = x_n$, $y = y_n$ in inequality (10), we obtain

$$\lim_{n \rightarrow \infty} \nu(\|x_n - y_n\|)\|x_n - y_n\| = 0,$$

whence it follows that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

In view of the equivalence of the norms $\|x\|$ and $m(x)$, from this we find that

$$\lim_{n \rightarrow \infty} m(x_n - y_n) = 0.$$

This proves the uniform convexity of the space E .

Let us note that, in fact, we have proved a stronger assertion: if in the Banach space E there exists a functional $f(x)$ satisfying the inequality

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) - f(\frac{1}{2}(x+y)) \geq c(\|x-y\|), \quad \forall x, y \in E,$$

where $c(t)$ is a continuous strictly increasing function of the nonnegative argument t , $c(0) = 0$, and

$$\lim_{t \rightarrow +\infty} c(t) = +\infty,$$

and if $f(x)$ is uniformly continuous on every bounded set in E , then there exists a norm in E , equivalent to the original one, in which the space E will be uniformly convex.

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