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Abstract

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MATHEMATICS

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ON BICOMPACTA WITH NONCOINCIDING DIMENSIONS

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In 1935 P. S. Aleksandrov raised the question whether, for bicompecta, the dimension defined by means of coverings ($\dim$) and the small inductive dimension (ind) coincide. A. G. Lund ⁽²⁾, and then O. V. Lokutsievskii ⁽¹⁾, constructed bicompecta that are one-dimensional in the sense of $\dim$ and two-dimensional in the sense of ind . At the same time, the bicompectum S of O. V. Lokutsievskii was considerably simpler than Lund's bicompectum; moreover, the bicompectum S decomposed into the sum of two such bicompecta S_1 and S_2 that $\text{ind } S_1 = \text{ind } S_2 = 1$. Thus, O. V. Lokutsievskii's example also showed that the sum theorem for the small inductive dimension does not hold in bicompecta. With the aid of a sufficiently general construction, P. Vopenka ⁽³⁾ constructed, for arbitrary natural numbers m and n ($1 \leq m < n$), such bicompecta X_m^n that $\dim X_m^n = m$, while $\text{ind } X_m^n = n$. All these bicompecta did not satisfy the first axiom of countability, and the dimensions there differed only on a nowhere dense set (namely, on the set of those points which do not have a countable fundamental system of neighborhoods).

Below we shall construct a separable bicompectum B with the first axiom of countability such that $\dim B = 2$ and $3 \leq \text{ind}_b B \leq 4$ for every point $b \in B$. One sufficiently general method for constructing examples of this kind will be indicated.

§ 1. Construction of the space B . Denote by J the half-interval $(0, 1]$ of the number line. Denote by I_n the segment $[1/(n+1), 1/n]$ of the number line. Then

$$J = \bigcup_{n=1}^{\infty} I_n.$$

Let $g : J \rightarrow T^2$ be such a continuous mapping of the half-interval J onto the torus T^2 (instead of the torus one may take any two-dimensional connected and locally connected compactum) that each segment I_n is mapped by g onto the

whole torus T^2 . Such a mapping g is easy to construct, starting, for example, from the standard Peano mapping of a segment onto a square.

Let now S^2 be the two-dimensional sphere of diameter 1 (it may be regarded as a subset of three-dimensional Euclidean space), and let x be some point of the sphere S^2 . Define a mapping $h_x : S^2 \setminus \{x\} \rightarrow T^2$ by the equality

$$h_x(x') = g(\rho(x, x')).$$

Here $\rho(x, x')$ denotes the distance from the point x to the point x' . The function ρ continuously maps the sphere with the point x removed, $S^2 \setminus \{x\}$, onto the half-interval J , and the mapping h_x is continuous as the superposition of the two continuous mappings ρ and g .

The fundamental property of the mapping h_x . For every nonempty open set $W \subset T^2$ and every neighborhood U of the point x , the set $U \cap h_x^{-1}W$ contains a closed set F which separates the sphere S^2 into such open sets D and E that $x \in D \subset U$.

Let $B = S^2 \times T^2$. We shall denote points of the set B by pairs (x, y) , where $x \in S^2$, $y \in T^2$. Define on the set B a topology τ , relative to which B will be the desired space. A base of the topology τ is formed by all sets of the form $O(U, x, W)$, where U is an open subset of the sphere S^2 , $x \in U$, W is an open subset of the torus T^2 , and

$$O(U, x, W) = (\{x\} \times W) \cup \{(U \cap h_x^{-1}W) \times T^2\} = W^x \cup U(W^x).$$

It is easy to verify that the family $\{O(U, x, W)\}$ forms a base for some topology.

Obviously, the topology τ is different from the product topology of S^2 on T^2 . At the same time the topology τ induces on each layer $\{x\} \times T^2$ the ordinary topology of the torus T^2 .

§ 2. Basic properties of the space B .

Theorem 1. *The space B is a separable bicom pactum with the first axiom of countability such that $\dim B = 2$ and $3 \leq \text{ind}_b B \leq 4$ for every point $b \in B$.*

Outline of the proof. 1. It is easy to verify that the space B is a separable bicom pactum with the first axiom of countability. Moreover, the projection $\pi : B \rightarrow S^2$ of the product $B = S^2 \times T^2$ onto the first factor is a continuous and irreducible mapping.

2. Let b be an arbitrary point of the bicom pactum B . The inequality $3 \leq \text{ind}_b B \leq 4$ follows from the following assertion.

Lemma. *Let G be an open subset of the bicom pactum B such that $B \setminus [G] \neq \emptyset$. Then there exists a point $x \in S^2$ such that $\{x\} \times T^2 = \pi^{-1}(x) \subset \text{Fr } G$.*

Proof. Denote by K the small image of the set G under the mapping π :

$$K = \pi^\# G = \{x \in S^2 \mid \pi^{-1}(x) \subset G\},$$

$$L = \pi^\#(B \setminus [G]), \quad M = S^2 \setminus K \cup L.$$

The sets K and L , by virtue of the irreducibility of the mapping π , are disjoint nonempty open subsets of the sphere S^2 . Hence the set M separates S^2 , and $\dim M = \text{ind } M \geq 1$. Let x be an arbitrary point of the set M for which $\text{ind}_x M \geq 1$. Using the basic property of the mapping h_x , it is easy to show that $\pi^{-1}(x) \subset \text{Fr } G$.

3. The proof of the inequality $\text{ind}_t B \leq 4$ reduces to the sufficiently cumbersome verification that the small inductive dimension of the boundary of any basic set $O(U, x, W)$ is at most 3. Whether in fact the small inductive dimension of the bicomcompactum B is equal to 3 or to 4 has not been established.
4. $\dim B = 2$. Since the space B contains, as a closed subset, a space homeomorphic to the two-dimensional torus, $\dim B \geq 2$. Let ω be an arbitrary cover of the bicomcompactum B . Since the mapping $\pi : B \rightarrow S^2$ is closed, there exists a finite set of points $x_1, \dots, x_l$ of the sphere S^2 such that the preimage $\pi^{-1}(x)$ of every point $x \neq x_j$, $j = 1, \dots, l$, is contained in some element of the cover ω together with the indicated neighborhood $\pi^{-1}(U_x)$. For each layer $\{x_j\} \times T^2 = \pi^{-1}(x_j)$, $j = 1, \dots, l$, one can choose a system β_j , consisting of basic sets of the space B and inscribed in the cover ω , such that the multiplicity of the system $\beta_j \leq 3$ and the body of the system β_j coincides with the set $U_j \times T^2 = \pi^{-1}U_j$, where U_j is a neighborhood of the point x_j . This can be done so that the sets U_j are pairwise disjoint and the multiplicity of the system

$$\beta = \bigcup_{j=1}^l \beta_j$$

in some neighborhood G of the set $B \setminus \bigcup_{j=1}^l \pi^{-1}U_j$ does not exceed one. There exists a system $\alpha = \{A_1, \dots, A_m\}$ of multiplicity ≤ 3 , consisting of open subsets of the sphere S^2 and covering the compactum $S^2 \setminus \bigcup_{j=1}^l U_j$, such that the system

$$\pi^{-1}(\alpha) = \{\pi^{-1}A_s \mid s = 1, \dots, m\}$$

is inscribed in the cover ω and has multiplicity ≤ 2 on the complement of the set $B \setminus \bigcup_{j=1}^l \pi^{-1}U_j$. Moreover, one may assume that the body of the system $\pi^{-1}(\alpha)$ is contained in the set G , where the multiplicity of the system β is not greater than one. Then the system $\beta \cup \pi^{-1}(\alpha)$ will be a cover of the bicomcompactum B , inscribed in the cover ω and having multiplicity ≤ 3 .

The bicom pactum B is not perfectly normal. It contains a subset of cardinality continuum which is discrete in the induced topology. The question remains open as to whether the dimensions $\dim$ and ind coincide for perfectly normal bicom pacta.

§ 3. General construction. Let X be an arbitrary bicom pactum, and let to each point $x \in X$ there be assigned a certain bicom pactum Y_x . Suppose that for each point $x \in X$ a continuous mapping $h_x : X \setminus \{x\} \rightarrow Y_x$ is defined. Then the system $\{X, Y_x, h_x \mid x \in X\}$ defines in the following way a bicom pactum B , continuously mapped onto the bicom pactum X by means of a projection $\pi : B \rightarrow X$ such that $\pi^{-1}(x) = Y_x$ for all $x \in X$: the bicom pactum B is defined on the set $\bigcup_{x \in X} Y_x$; a base for the topology is formed by all possible sets of the form

$$O(U, x, W) = W \cup \pi^{-1}(U \cap h_x^{-1}W),$$

where W is an open subset of the bicom pactum Y_x , and U is a neighborhood of the point x in the bicom pactum X . We shall denote the bicom pactum B by $B\{X, Y_x, h_x\}$. In what follows we shall always assume that the functions h_x satisfy the following property: for every nonempty open set $W \subset Y_x$ and every neighborhood U of the point x , the set $U \cap h_x^{-1}W$ contains a closed set F which partitions the space X into such open sets D and E that $x \in D \subset U$.

Theorem 2. If $\text{Ind } X \leq n$ and $\dim Y_x \leq n$ for all $x \in X$, then

$$\dim B\{X, Y_x, h_x\} \leq n.$$

Theorem 3. If $\dim X \geq n$ and $\text{ind } Y_x \geq n$ for all $x \in X$, then

$$\text{ind } B\{X, Y_x, h_x\} \geq 2n - 1.$$

Theorem 4. If the bicom pactum X is hereditarily normal, $\text{Ind } X \leq n$, and $\text{ind } Y_x \leq n$ for all $x \in X$, then

$$\text{ind } B\{X, Y_x, h_x\} \leq 2n.$$

Corollary. If the bicom pactum X is hereditarily normal, $\dim X = \text{Ind } X = n$, and $\text{ind } Y_x = n$ for all $x \in X$, then

$$\dim B \leq n \quad \text{and} \quad 2n - 1 \leq \text{ind } B \leq 2n.$$

Remark. There exist families $\{X, Y_x, h_x\}$ satisfying all the conditions of this paragraph. Indeed, one such family is the following: the space X coincides with the n -dimensional sphere S^n , the spaces Y_x for all $x \in X$ coincide with the n -dimensional torus T^n (instead of T^n one may take an arbitrary n -dimensional connected and locally connected compactum), and the mappings $h_x : S^n \setminus \{x\} \rightarrow$

T^n are obtained from a mapping $g : J \rightarrow T^n$ in the same way as, in the first paragraph, the mappings $h_x : S^2 \setminus \{x\} \rightarrow T^2$ are obtained from the mapping $g : J \rightarrow T^2$.

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