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1968

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Abstract

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UDC 517.946.9+517.946.82

MATHEMATICS

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A VARIATIONAL METHOD FOR SOLVING IRREGULAR ELLIPTIC EQUATIONS

(Presented by Academician I. M. Vinogradov on 17 V 1968)

We study the equation

$$Au = (-1)^l \sum_{\substack{|\alpha|=l \\ |\beta|=l}} D^\beta [a_{\alpha\beta}(x) D^\alpha u(x)] = f$$

in an arbitrary open set O of the n -dimensional Euclidean space E_x^n of variables $x = (x_1, \dots, x_n)$, where α, β are n -dimensional vectors with integer nonnegative coordinates, and $|\alpha|, |\beta|$ are the sums of these coordinates. As the domain of definition $\Omega(A)$ of the operator A we take finite real-valued functions $u = u(x)$ having generalized square-summable derivatives $D^\alpha u(x)$ of order l such that the functions $a_{\alpha\beta}(x) D^\alpha u(x)$, in turn, have square-summable generalized derivatives $D^\beta [a_{\alpha\beta}(x) D^\alpha u(x)]$, $|\beta| = l$. The operator A is assumed to be nonnegative and symmetric,

$$(Au, u) = \int_O Au \cdot u \, dx \geq 0, \quad (Au, v) = (u, Av)$$

for all $u, v \in \Omega(A)$.

Let H be the Hilbert space that is the completion of $\Omega(A)$ in the metric defined by the scalar product $[u, v] = (Au, v)$, and let H^* be its dual.

A generalized solution of the equation $Au = f$, $f \in H^*$, is a function $u_0 \in H$ such that $[u_0, v] = f(v)$ for all $v \in H$. The variational method of solving the equation $Au = f$ consists in finding an element of the space H that realizes the minimum of the functional $F(u) = [u, u] - 2f(u)$. As is known, the generalized solution always exists and is unique in H , and can be found by the variational method.

Let $a = a(x)$ be a measurable almost everywhere positive function,

$$D^l u = \left[\sum_{i_1, \dots, i_l=1}^n \left[\frac{\partial^l u(x)}{\partial x_{i_1} \dots \partial x_{i_l}} \right]^2 \right]^{1/2},$$

$1 < p < \infty$. We define the space $L_{p,a}^{(l)}(O)$ as the completion of finite functions having generalized derivatives of order l , in the norm

$$\|u, L_{p,a}^{(l)}(O)\| = \left[\int_O a(x) |D^l u|^p dx \right]^{1/p}.$$

In the case $l = 0$ we shall write $L_{p,a}^{(0)}(O) \equiv L_{p,a}(O)$.

The operator A is called elliptic if there exists a measurable almost everywhere positive function $a(x)$ such that, for all $u \in \Omega(A)$,

$$(Au, u) \geq \int_O a(x) |D^l u|^2 dx.$$

Thanks to the embedding $H \subset L_{2,a}^{(l)}(O)$, for an elliptic operator A the investigation of the solution—its differential properties, satisfaction of boundary conditions, etc.—is reduced to the study of the space $L_{2,a}^{(l)}(O)$.

In the proof of theorems of the type $L_{p,a}^{(l)}(O) \subset L_{p,b}(O)$, generalizations of the known Hardy inequalities are used. Let $u(x)$ and $a(x)$ be nonnegative measurable functions on the real axis, M and N numbers, $-\infty \leq M < N \leq +\infty$, and l a natural number. Introduce the notation

$$\langle l \rangle \int_M^x u(t) dt = \int_M^x dt_{l-1} \int_M^{t_{l-1}} dt_{l-2} \dots \int_M^{t_1} u(t) dt.$$

Similarly, $\langle l \rangle \int_x^N u(t) dt$ is defined.

Lemma. The inequalities

$$\begin{aligned} & \int_M^N (N-x)^{(l-1)p/(p-1)} \left[\int_M^x (N-t)^{(l-1)p/(p-1)} a^{1/(1-p)}(t) dt \right]^{-p} \left[\langle l \rangle \int_M^x u(t) dt \right]^p dx \leq \\ & \leq \frac{1}{[(l-1)!]^p} \left(\frac{p}{p-1} \right)^p \int_M^N a(x) u^p(x) dx, \quad -\infty \leq M < N < +\infty; \end{aligned}$$

$$\begin{aligned} & \int_M^N (x-M)^{-(l-1)p} a^{1/(1-p)}(x) \left[\int_M^x a^{1/(1-p)}(t) dt \right]^{-p} \left[\langle l \rangle \int_M^x u(t) dt \right]^p dx \leq \\ & \leq \frac{1}{[(l-1)!]^p} \left(\frac{p}{p-1} \right)^p \int_M^N a(x) u^p(x) dx, \quad -\infty < M < N \leq +\infty, \end{aligned}$$

and the analogous ones for estimating $\langle l \rangle \int_x^N u(t) dx$. For $l = 1$ both limits of integration may be infinite.

Other generalizations of Hardy inequalities were obtained by V. R. Portnov ⁽¹⁾ and F. A. Sysoeva ⁽²⁾.

Suppose that in the open set O a system of coordinates is introduced which maps O one-to-one onto an open set $\tilde{O}$ of the n -dimensional Euclidean space of variables (r, γ) , where r is the distinguished variable and γ is an $(n-1)$ -dimensional vector. The functions $x_i(r, \gamma)$, $i = 1, 2, \dots, n$, will be assumed l times continuously differentiable in the closure of the set $\tilde{O}$, and the Jacobian $D(x)/D(r, \gamma)$ positive in $\tilde{O}$.

Passing in the integral $\int_O a(x) |D^l u|^p dx$ to the variables (r, γ) and using the generalized Hardy inequalities, one can always find a weight $b(x)$ such that

$$\int_O b(x) |u(x)|^p dx \leq \int_O a(x) |D^l u|^p dx.$$

With a sufficiently strong degeneration of the weight $a(x)$, the elements of $L_{p,a}^{(l)}(O)$, generally speaking, need not be locally summable functions. The weight $b(x)$ depends essentially on the geometric properties of the set in every neighborhood of whose points the function $a^{1/(1-p)}(x)$ is nonintegrable. For example, one can give an example of a weight $a(x)$ for which $a^{1/(1-p)}(x)$ is nonintegrable in a neighborhood of the points of the surface of a ball lying in O , and there exists a function belonging to $L_{p,a}^{(l)}(O)$ and identically equal to $+\infty$ inside the ball. The corresponding weight $b(x)$ inside the ball is equal to zero.

We investigate the question of preservation of boundary conditions for functions from the class $L_{p,a}^{(l)}(O)$. Boundary values are understood in the sense of the limit of the function almost everywhere along the given field of directions r . Moreover, if one is speaking of boundary values of derivatives of the function, derivatives with respect to r are meant. This definition is applicable to a set O such that $\text{mes } \Pi_r \partial \tilde{O} > 0$, where $\Pi_r E$ denotes the projection of the set $E \subset E_{(r,\gamma)}^n$ onto the hy-

the hyperplane Γ of the vectors γ , and $\partial\tilde{O}$ is the boundary of $\tilde{O}$. We shall assume that the indicated condition is satisfied for each connected component of the set O . For questions connected with boundary conditions at infinity for unbounded domains, including the whole of E^n , see (3).

In a certain sense one may also speak of boundary values in the mean.

Let us also denote by $\bar{\partial}G(\underline{\partial}G)$ the upper (lower) semiboundary of an open set $G \subset E_{(r,\gamma)}^n$, i.e., those boundary points which are the upper (lower) endpoints of the intervals obtained by intersecting G with the straight lines $\gamma = \text{const}$.

Theorem 1. *Let there exist a finite or countable system $\{O_i\}$ of open sets $O_i \subset O$ such that*

$$\text{mes } \Pi_\Gamma \left(\partial\tilde{O} \setminus \bigcup_i \partial\tilde{O}_i \right) = 0$$

and

$$\text{mes } \Pi_\Gamma \left(\partial\tilde{O} \setminus \bigcup_i \partial\tilde{O}_i \right) = 0,$$

and for each O_i

$$\int_{O_i} \left[\frac{D(x)}{D(r, \gamma)} \right]^{-p/(p-1)} a^{1/(1-p)}(x) dx < \infty,$$

while the field of directions satisfies the condition $\partial^2 x / \partial r^2 = 0$. Then every function $u(x) \in L_{p,a}^{(l)}(O)$, together with its derivatives up to order $l-1$ inclusive, is equal to zero on the boundary of O both in the mean and in the sense of almost everywhere.

Let us note that here $u(x)$ need not be a locally summable function not only inside O , but also in an arbitrarily small neighborhood of its boundary.

Theorem 2. *Let the inequality*

$$\int \left[\frac{D(x)}{D(r, \gamma)} \right]^{-p/(p-1)} a^{1/(1-p)}(x) dx < \infty$$

be satisfied for all of O or for any finite part of it, and let the condition on the field of directions be as before. Then the function $u(x) \in L_{p,a}^{(l)}(O)$ has all generalized derivatives up to order l inclusive. If, moreover,

$$a^{1/(1-p)}(x) \in L_s^{\text{loc}}(O)$$

for some s , $1 \leq s \leq \infty$, then the generalized derivatives of order l belong to

$$L_{\frac{p}{1+(p-1)/s}}^{\text{loc}}(O).$$

The requirement $\partial^2 x / \partial r^2 = 0$ is inessential and was introduced for brevity of the formulas. For a general field of directions we restrict ourselves to $l = 1$, bearing in mind that one can use embedding theorems of the type

$$L_{p,a}^{(l)}(O) \subset L_{p,b_k}^{(k)}(O).$$

Theorem 3. *Let, in O or in any finite part of it,*

$$\int \left[\frac{D(x)}{D(r, \gamma)} \right]^{-p/(p-1)} a^{1/(1-p)}(x) dx < \infty.$$

Then the function $u(x) \in L_{p,a}^{(l)}(O)$ has first generalized derivatives and assumes on the boundary the value zero almost everywhere and in the mean. If, moreover,

$$a^{1/(1-p)}(x) \in L_s^{\text{loc}}(O)$$

for some s , $1 \leq s \leq \infty$, then the first generalized derivatives belong to

$$L_{\frac{p}{1+(p-1)/s}}^{\text{loc}}(O).$$

Let us return to the equation $Au = f$.

Theorem 4. Suppose that the conditions of Theorem 2 or 3 are satisfied with $p = 2$. Then, for the elliptic equation $Au = f$, there exists a generalized solution, unique up to a set of measure zero, for any right-hand side $f \in H^*$. This solution belongs to the space $L_{2,a}^{(l)}(O)$, has generalized derivatives up to order l inclusive, and satisfies the zero boundary conditions almost everywhere and, on the average, together with its derivatives up to order $l - 1$ inclusive. If, in addition, the coefficients of the equation

$$a_{\alpha\beta}(x) \in L_{\frac{2}{1-1/s}}^{\text{loc}}(O)$$

(where s is the same as in the conditions of Theorem 2 or 3), then the equation is satisfied in the following sense: $a_{\alpha\beta}(x)D^\alpha u(x)$ are locally summable functions and have, in the sense of the theory of generalized functions, derivatives $D^\beta(a_{\alpha\beta}(x)D^\alpha u(x))$, the sum of which is equal to f .

We now consider the equation $Au = f$ with right-hand side a function $f = f(x) \in L_{2,1/b}(O)$. It is assumed that $b(x) > 0$ almost everywhere. The embedding $L_{2,1/b}(O) \subset H^*$ holds.

Theorem 5. For any function $f(x) \in L_{2,1/b}(O)$, the elliptic equation $Au = f$ has, and moreover uniquely, a generalized solution in the space $L_{2,a}^{(l)}(O)$. The resulting extension of the operator

$$\tilde{A} = \frac{1}{b(x)}A,$$

acting from $L_{2,b}(O)$ to $L_{2,b}(O)$, is a self-adjoint operator with positive (equal to one) lower bound.

Under stronger restrictions on the coefficients and the domain, similar questions were also studied earlier (see, for example, ^(4–8)). For unbounded domains, the variational method was extended by L. D. Kudryavtsev ^(9, 10); his results were developed by V. R. Portnov ^(1, 11), Yu. S. Nikol'skii ^(12, 13), and T. S. Pitolkina ⁽¹⁴⁾.

In conclusion, the author expresses sincere gratitude to Prof. L. D. Kudryavtsev for his supervision of the work.

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Received

14 V 1968

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