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1968

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Abstract

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UDC 539.3

THEORY OF ELASTICITY

A. N. GUZ

ON THE ACCURACY OF THE KIRCHHOFF-LOVE HYPOTHESIS IN DETERMINING CRITICAL FORCES IN THE THEORY OF ELASTIC STABILITY

(Presented by Academician A. Yu. Ishlinskii, 19 V 1967)

To clarify the accuracy of determining critical forces by applied theories based on the use of the Kirchhoff-Love hypothesis, let us consider the stability of elastic systems from the point of view of the three-dimensional linearized equations of the nonlinear mathematical theory of elasticity ⁽⁴⁾. To shorten the exposition, we shall investigate the stability of a thick orthotropic plate under a cylindrical form of buckling independent of the coordinate z , when the plate is compressed by forces of intensity P along the ox axis (Fig. 1). It is assumed that the plate is infinite along the ox axis, or else that the boundary conditions for $x = \text{const}$ are satisfied in an integral sense; all coordinates are dimensionless, referred to h , the half-thickness of the plate.

Fig. 1.

The basic equations may be written in the form

$$\begin{aligned} \left(a_{11} \frac{\partial^2}{\partial x^2} + G \frac{\partial^2}{\partial y^2} \right) u + (G + a_{12}) \frac{\partial^2}{\partial x \partial y} v &= 0; \\ \left[(G - P) \frac{\partial^2}{\partial x^2} + a_{22} \frac{\partial^2}{\partial y^2} \right] v + (G + a_{12}) \frac{\partial^2}{\partial x \partial y} u &= 0. \end{aligned} \quad (1)$$

The stresses are determined by the formulas

$$\begin{aligned} h\sigma_x &= a_{11} \partial u / \partial x + a_{12} \partial v / \partial y; & h\sigma_y &= a_{12} \partial u / \partial x + a_{22} \partial v / \partial y; \\ h\tau_{xy} &= G(\partial u / \partial y + \partial v / \partial x). \end{aligned} \quad (2)$$

In deriving equations (1), it was assumed that, in calculating the angles of rotation, the strains may be taken to be approximately equal to zero.

It can be shown that the general solution of equations (1) is

$$v = \left(a_{11} \frac{\partial^2}{\partial x^2} + G \frac{\partial^2}{\partial y^2} \right) \psi; \quad u = -(G + a_{12}) \frac{\partial^2}{\partial x \partial y} \psi; \quad \psi = \psi_1 + \psi_2;$$

$$\left(\frac{\partial^2}{\partial y^2} + \xi_i^2 \frac{\partial^2}{\partial x^2} \right) \psi_i = 0; \quad i = 1, 2; \quad \xi_{1,2}^2 = \frac{a_{11}a_{22} - a_{12}^2 - 2a_{12}G}{2a_{22}G} - \frac{1}{2} \frac{a_{11}}{a_{22}} p \pm \pm \sqrt{\left(\frac{a_{11}a_{22} - a_{12}^2 - 2a_{12}G}{2a_{22}G} - \frac{1}{2} \frac{a_{11}}{a_{22}} p \right)^2 + \left(\frac{a_{11}}{a_{22}} \right)^2}.$$

The boundary conditions on the free surface have the form

$$\left(\frac{\partial^2}{\partial y^2} + \frac{a_{11}a_{22} - a_{12}^2 - a_{12}G}{a_{22}G} \frac{\partial^2}{\partial x^2} \right) \frac{\partial \psi}{\partial y} \Big|_{y=\pm 1} = 0; \quad \left(\frac{\partial^2}{\partial y^2} - \frac{a_{11}}{a_{12}} \frac{\partial^2}{\partial x^2} \right) \frac{\partial \psi}{\partial x} \Big|_{y=\pm 1} = 0. \quad (4)$$

A solution even in y for v is written in the form

$$\psi = \{A \operatorname{ch} \alpha \xi_1 y + B \operatorname{ch} \alpha \xi_2 y\} \cos \alpha x; \quad \alpha = \pi h/l. \quad (5)$$

We note that for $x = \pm l/2h$ the conditions of hinged support are satisfied; these conditions are also preserved in the transition to a thin-walled structure. Thus, the limiting case of the problem under consideration is the stability of a hinged thin plate.

Substituting the solution (5) into the boundary conditions (4), from the condition for the existence of nontrivial solutions we obtain the transcendental equation for determining p

$$\left(\xi_1^2 - \frac{a_{11}a_{22} - a_{12}^2 - a_{12}G}{a_{22}G} \right) \left(\xi_2^2 + \frac{a_{11}}{a_{12}} \right) \xi_1 \operatorname{sh} \alpha \xi_1 \operatorname{ch} \alpha \xi_2 - \left(\xi_2^2 - \frac{a_{11}a_{22} - a_{12}^2 - a_{12}G}{a_{22}G} \right) \left(\xi_1^2 + \frac{a_{11}}{a_{12}} \right) \xi_2 \operatorname{sh} \alpha \xi_2 = 0.$$

Let us analyze the roots of equation (6) for thin-walled structures; in this case $\alpha \ll 1$. Accordingly, in the hyperbolic functions we retain the first three terms. As a result we obtain

$$\left(\xi_1^2 \xi_2^2 - \frac{a_{11}}{a_{12}} \frac{a_{11}a_{22} - a_{12}^2 - a_{12}G}{a_{22}G} \right) \times \left[1 + \alpha^2 \frac{\xi_1^2 + \xi_2^2}{6} + \alpha^4 \frac{(\xi_1^2 + \xi_2^2)^2 + 4\xi_1^2 \xi_2^2}{120} \right] + \frac{a_{11}a_{22} - a_{12}^2 - a_{12}G}{a_{22}G} - \xi_1^2 \xi_2^2 = 0.$$

From relations (3) we determine $\xi_1^2 + \xi_2^2$ and $\xi_1^2 \xi_2^2$:

$$\xi_1^2 \xi_2^2 = \frac{a_{11}}{a_{22}} - \frac{a_{11}^2}{a_{22} G} p; \quad \xi_1^2 + \xi_2^2 = \frac{a_{11} a_{22} - a_{12}^2 - 2a_{12} G}{a_{22} G} - \frac{a_{11}}{a_{22}} p. \quad (8)$$

Substituting expressions (8) into equation (7), we obtain a cubic equation for determining p . We note that in elastic stability $p \ll 1$; therefore, substituting p in the form of a series in α , we determine, to accuracy up to α^4 ,

$$P_{cr} \simeq P_{el} \left[1 - \alpha^2 \frac{2}{15} \frac{3(a_{11} a_{22} - a_{12}^2) - 2a_{12} G}{a_{22} G} \right], \quad (9)$$

where P_{el} is the value of the critical force obtained using the Kirchhoff-Love hypothesis (1):

$$P_{el} \simeq \frac{\alpha^2}{3} \frac{a_{11} a_{22} - a_{12}^2}{a_{11}}. \quad (10)$$

Thus, the first term of the expansion, for small α , of the critical force computed by the exact theory coincides with the value of the critical force computed using the Kirchhoff-Love hypothesis, i.e., for thin-walled structures made of isotropic materials one may apply the Kirchhoff-Love hypothesis.

Expressing a_{11} , a_{22} , and a_{12} in terms of Young's moduli and Poisson's ratios, one can arrive at the conclusion that, for thin-walled structures made of orthotropic materials, the Kirchhoff-Love hypothesis may be used in computing critical forces when

$$\alpha^2 \ll \frac{5}{2} \frac{G}{E_1}. \quad (11)$$

A. Yu. Ishlinskii (2) considered this same problem for an isotropic body, proceeding from the Lamé equations. He came to the conclusion that the Kirchhoff-Love hypothesis may be applied when computing the critical forces for thin-walled structures. I. D. Legen' arrived at the opposite conclusion (3); the inaccuracy of his assertion was caused by the fact that boundary conditions were used which do not coincide with the boundary conditions of the work (4).

Consider the value of P_{cr} (9) for an isotropic body; in this case

$$a_{11} \equiv a_{22} = \lambda + 2\mu; \quad a_{12} = \lambda; \quad G = \mu; \quad P_{cr} \approx P_{el} \left(1 - \alpha^2 \frac{2}{15} \frac{6 - \nu}{1 - \nu} \right). \quad (12)$$

As an example, consider the case $\nu = 0.3$. From formula (12) we obtain

$$P_{\text{cr}} \approx P_{\text{el}}(1 - 1.0857 \alpha^2), \quad (13)$$

and from the formula of paper ²

$$P_{\text{cr}} \approx P_{\text{el}}(1 + 0.0095 \alpha^2). \quad (14)$$

Institute of Mechanics
Academy of Sciences of the USSR

Received
27 IV 1967

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Note: Figure translations are in progress. See original paper for figures.

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