

ON SOLVABLE GROUPS OF FINITE RANK

D. I. Zaitsev

1968

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196801.74168>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.41/47

MATHEMATICS

D. I. ZAITSEV

ON SOLVABLE GROUPS OF FINITE RANK

(Presented by Academician V. M. Glushkov on 16 X 1967)

A rational series of a group $\mathfrak{G}$ is an ascending normal series of this group

$$\mathfrak{R}_0 = E \subset \mathfrak{R}_1 \subset \dots \subset \mathfrak{R}_\alpha \subset \mathfrak{R}_{\alpha+1} \subset \dots \subset \mathfrak{R}_\gamma = \mathfrak{G},$$

whose factors $\mathfrak{R}_{\alpha+1}/\mathfrak{R}_\alpha$ are isomorphic to certain subgroups of the additive group of rational numbers ⁽¹⁾.

For a group $\mathfrak{G}$ possessing a rational series, the notion of rational rank is introduced. Namely, if $\mathfrak{G}$ has a rational series of finite length k , then the rational rank of $\mathfrak{G}$ is taken to be k ; otherwise the rational rank of the group is considered infinite. We shall denote the rational rank of the group $\mathfrak{G}$ by $r(\mathfrak{G})$. The symbol $s(\mathfrak{G})$ denotes the special rank ⁽²⁾ of the group $\mathfrak{G}$.

The question naturally arises of the relation between $r(\mathfrak{G})$ and $s(\mathfrak{G})$. In this direction V. M. Glushkov ⁽³⁾ established that in the case of a locally nilpotent group $\mathfrak{G}$ the equality $r(\mathfrak{G}) = s(\mathfrak{G})$ holds. In the present article it is reported (Theorem 2) that, for any group possessing a rational series, the special rank is equal to the rational rank.

The proof of the main result is based on a number of properties of solvable groups satisfying the weak minimal condition for subgroups. Let us recall the definition.

A group $\mathfrak{G}$ satisfies the weak minimal condition for subgroups if there does not exist in it an infinite descending chain of subgroups

$$\mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \dots \supset \mathfrak{G}_k \supset \mathfrak{G}_{k+1} \supset \dots,$$

satisfying the following condition: the index $[\mathfrak{G}_k : \mathfrak{G}_{k+1}]$ of the subgroup $\mathfrak{G}_{k+1}$ in the subgroup $\mathfrak{G}_k$ is infinite, $k = 1, 2, \dots$. Groups satisfying this minimal condition were studied by the author in ⁽⁴⁾.

We now formulate an auxiliary proposition, which gives a condition that singles out, from the class of nilpotent groups of finite rank, the class of nilpotent groups satisfying the weak minimal condition for subgroups.

Lemma 1. *Let there exist in the nilpotent group of finite rank $\mathfrak{G}$ a system of generating elements $X_1, X_2, \dots, X_k$, $\mathfrak{M}$, and such a natural number n that for every element $Y \in \mathfrak{M}$ there is an l for which*

$$Y^{n^l} \in \{X_1, X_2, \dots, X_k\}.$$

Then $\mathfrak{G}$ satisfies the weak minimal condition for subgroups, and for every $G \in \mathfrak{G}$ one can specify a number m for which

$$G^{n^m} \in \{X_1, X_2, \dots, X_k\}.$$

Conversely, if a nilpotent group $\mathfrak{G}$ satisfies the weak minimal condition for subgroups, then in $\mathfrak{G}$ there exists such a finite system of elements $X_1, X_2, \dots, X_k$ and such a natural number n that for every element $G \in \mathfrak{G}$ one can specify a number m for which $G^{n^m} \in \{X_1, X_2, \dots, X_k\}$.

Corollary 1. *In a nilpotent group of finite rank $\mathfrak{G}$, a subgroup generated by a finite set of subgroups satisfying the weak minimal condition also satisfies the weak minimal condition.*

Corollary 2. *In a locally nilpotent torsion-free group $\mathfrak{G}$, a subgroup generated by a finite set of subgroups satisfying the weak minimal condition also satisfies the weak minimal condition.*

This follows from Theorem 4 of paper ⁽³⁾ and Corollary 1 of Lemma 1. Let us note that for a periodic locally nilpotent group, Corollary 2 does not hold. A counterexample can be found in the paper of O. Yu. Schmidt ⁽⁵⁾.

One characterization of nilpotent torsion-free groups of finite rank is given by

Lemma 2. *A nilpotent torsion-free group $\mathfrak{R}$ has finite rank if and only if there does not exist in $\mathfrak{R}$ an infinite system of elements $X_1, X_2, \dots, X_i, \dots$ having the following property:*

$$\{X_1, X_2, \dots, X_i\} \cap \{X_{i+1}\} = E, \quad i = 1, 2, \dots$$

Lemma 3. *Let φ be an automorphism of a nilpotent group of finite rank $\mathfrak{R}$, and let $\mathfrak{L}$ be a subgroup of $\mathfrak{R}$ satisfying the weak minimality condition for subgroups. Then the least subgroup $\mathfrak{L}^*$, invariant with respect to φ and containing $\mathfrak{L}$, also satisfies the weak minimality condition for subgroups.*

Lemma 4. *If $\mathfrak{G}$ is a nilpotent group of finite rank and the factor group $\mathfrak{G}/\mathfrak{G}'$ of the group $\mathfrak{G}$ by its commutator subgroup $\mathfrak{G}'$ satisfies the weak minimality condition for subgroups, then $\mathfrak{G}$ also satisfies the weak minimality condition for subgroups.*

Relying on the formulated lemmas and the results of the work of A. I. Mal'cev ⁽⁶⁾, one can establish the following theorem, which is of definite interest for the theory of solvable groups of finite rank.

Theorem 1. *A solvable torsion-free group of finite rank with a finite number of generators satisfies the weak minimality condition for subgroups.*

Lemma 5. *Let a group $\mathfrak{G}$ possess a rational series of length k and satisfy the weak minimality condition for subgroups. Then in $\mathfrak{G}$, for every sufficiently large prime number p , there exist two characteristic subgroups of finite index $\mathfrak{A}, \mathfrak{B}$, such that $\mathfrak{A} \supset \mathfrak{B}$ and the factor group $\mathfrak{A}/\mathfrak{B}$ is an elementary abelian p -group of order p^k .*

From Lemma 5 and Theorem 1 there immediately follows the main result of the present work.

Theorem 2. *If $\mathfrak{G}$ is a group possessing a rational series, then the special rank of $\mathfrak{G}$ is equal to its rational rank, i.e. $s(\mathfrak{G}) = r(\mathfrak{G})$.*

We indicate some known results obtained as consequences of Theorem 2.

Corollary 1. (V. M. Glushkov ⁽³⁾). *In order that a locally nilpotent torsion-free group be a nilpotent group of finite special rank k , it is necessary and sufficient that it possess a rational series of length k .*

Corollary 2. (S. N. Chernikov ⁽¹⁾, B. I. Plotkin ⁽⁷⁾). *If a group possesses a finite rational series, then the length of such a series is an invariant of the group. The length of a rational series of a proper isolated subgroup is strictly less than the length of a rational series of the whole group.*

Corollary 3. (B. I. Plotkin ⁽⁸⁾). *Let a group $\mathfrak{G}$ possess an ascending rational series. In this case, in order that $\mathfrak{G}$ have finite rank, it is necessary and sufficient that $\mathfrak{G}$ possess finite special rank.*

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
13 X 1967

REFERENCES

- ¹ S. N. Chernikov, *Uch. zap. Ural' skogo gos. univ.*, vol. 7, 3 (1949).
- ² A. I. Mal' cev, *Matem. sborn.*, 22, 351 (1948).
- ³ V. M. Glushkov, *Matem. sborn.*, 30, 79 (1952).
- ⁴ P. I. Zaitsev, *DAN*, 178, No. 4 (1968).
- ⁵ O. Yu. Schmidt, *Matem. sborn.*, 8, 363 (1940).
- ⁶ A. I. Mal' cev, *Matem. sborn.*, 28, 567 (1951).
- ⁷ B. I. Plotkin, *Matem. sborn.*, 30, 197 (1952).
- ⁸ B. I. Plotkin, *Tr. Mosk. matem. obshch.*, 6, 299 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.