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I. S. Shikin

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Abstract

Full Text

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PHYSICS

I. S. SHIKIN

SOLUTION OF THE GRAVITATIONAL EQUATIONS IN A HOMOGENEOUS, COMPLETELY ANISOTROPIC MODEL

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In this article, solutions of the equations of the general theory of relativity are considered for the case of a homogeneous but completely anisotropic 3-space in the presence of matter. These anisotropic solutions (with some exceptions) asymptotically tend to the isotropic Friedmann solution in the flat model; however, the anisotropy manifests itself essentially near singularities. A special case of solutions of the type considered is the flat model with axial symmetry ⁽²⁻⁴⁾.

A completely anisotropic solution for dust-like matter was given in ⁽¹⁾. Below, the corresponding formulas are given for an ultrarelativistic equation of state.

1. As the coordinate system, a comoving system is used, in which the medium is at rest. This comoving system is synchronous (so that the lines of proper time τ , coinciding with the world lines of the "fluid" particles, are geodesics of 4-space). The metric in the model under consideration has the form

$$-ds^2 = -(c d\tau)^2 + [R_1(\tau)dx^1]^2 + [R_2(\tau)dx^2]^2 + [R_3(\tau)dx^3]^2. \quad (1)$$

The functions $R_1(\tau), R_2(\tau), R_3(\tau)$ are determined by the Einstein gravitational equations (the notation of ⁽⁵⁾ is used)

$$R_i^k = (8\pi k/c^4) [T_i^k - (T/2)\delta_i^k]. \quad (2)$$

In the metric (1) the 3-space x^1, x^2, x^3 is Euclidean; however R_i^k differ from 0, and the 4-space is curved.

In equations (2), T_i^k is the energy-momentum tensor of an ideal gas. In the comoving system used, T_i^k has diagonal form: $T_1^1 = T_2^2 = T_3^3 = p$, where p is the pressure; $-T_0^0 = e$, where e is the density of internal energy; $T = 3p - e$; p and e depend on τ .

The determinant of the metric tensor $(-g)$ is equal to

$$(-g)^{1/2} = R_1(\tau)R_2(\tau)R_3(\tau). \quad (3)$$

The nonzero components of equations (2) have the form ⁽⁵⁾ (a dot denotes differentiation with respect to τ):

$$R_0^0 : \frac{1}{c^2} \frac{d^2}{d\tau^2} \ln \sqrt{-g} + \frac{1}{c^2} \left(\frac{\dot{R}_1^2}{R_1^2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{\dot{R}_3^2}{R_3^2} \right) = -\frac{8\pi k}{c^4} \frac{e + 3p}{2}, \quad (4)$$

$$R_1^1, R_2^2, R_3^3 : \frac{1}{c^2 \sqrt{-g}} \frac{d}{d\tau} \left[\sqrt{-g} \frac{d}{d\tau} \ln R_\alpha \right] = \frac{8\pi k}{c^4} \frac{e - p}{2}, \quad \alpha = 1, 2, 3. \quad (5)$$

The equations $T_{i;k}^k = 0$ contained in (2) give ($i = 0$)

$$-\dot{e}/(\dot{e} + p) = (-\dot{g})/2(-g). \quad (6)$$

Adding the three equations (5), we obtain an equation for $(-g)$

$$d^2[(-g)^{1/2}]/d\tau^2 = (12\pi k/c^2)(e - p)(-g)^{1/2}. \quad (7)$$

Equations (6) and (7) are conveniently used to determine, for a given equation of state, the dependence $-g(\tau)$, after which R_1, R_2 , and R_3 are determined from (4) and (5).

In the model (1) under consideration, the three Hubble “constants” $h_\beta = d \ln R_\beta / d\tau$, $\beta = 1, 2, 3$, satisfy the relation obtained after substituting h_β into the equation $R_0^0 - (R'/2) = (8\pi k/c^4)T_0^0$

$$h_1 h_2 + h_1 h_3 + h_2 h_3 = 8\pi k e / c^2. \quad (8)$$

2. Dust-like matter ($p = 0$, $e = \mu c^2$, μ is the mass density)

From (6) it follows that $e = \text{const}/(-g)^{1/2}$ (conservation of mass in a “liquid” volume), and from (7) it follows that

$$(-g)^{1/2} = Lc^2(\tau - \tau_1)(\tau - \tau_2), \quad L = \text{const}, \quad \tau_1 = \text{const}, \quad \tau_2 = \text{const}. \quad (9)$$

The formal possibility of complex conjugate values τ_1 and τ_2 in (9) does not lead to real values for R_1, R_2 , and R_3 , and is not physically realized. This is connected with the fact that the determinant $(-g)$ does not vanish, which is impossible in a synchronous system ^(5,6).

From equations (5), in consequence of (9), it follows that

$$\begin{aligned}(R_1)^3 &= L_1 c^2 (\tau - \tau_1)^{1+\alpha_1} (\tau - \tau_2)^{1-\alpha_1}, \\ (R_2)^3 &= L_2 c^2 (\tau - \tau_1)^{1+\alpha_2} (\tau - \tau_2)^{1-\alpha_2}, \\ (R_3)^3 &= L_3 c^2 (\tau - \tau_1)^{1+\alpha_3} (\tau - \tau_2)^{1-\alpha_3}.\end{aligned}\tag{10}$$

L_1, L_2, L_3 are constants having the meaning of coordinate scales; $L_1 L_2 L_3 = L^3$.

The quantities $\alpha_1, \alpha_2, \alpha_3$ in (10) are constants that are essential in the solution. They are not completely arbitrary, but are connected by two relations. One is a consequence of (9):

$$\alpha_1 + \alpha_2 + \alpha_3 = 0.\tag{11}$$

The second relation is obtained after substituting (10) into (4):

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 6.\tag{12}$$

From (11) and (12) it follows that, for a given value of α_1 , the values α_2 and α_3 are roots of the equation $\alpha^2 + \alpha\alpha_1 + (\alpha_1^2 - 3) = 0$. These roots are real for $\alpha_1^2 \leq 4$, so that α_1 , and likewise α_2 and α_3 , do not exceed 2 in absolute value. As α_1 increases monotonically in the interval from -1 to 1 , the values α_2 and α_3 decrease monotonically: one from 2 to 1 , the other from -1 to -2 , together with α_1 covering the whole domain of admissible values. Therefore, without loss of generality, the coordinates may be considered numbered so that

$$-1 \leq \alpha_1 \leq 1, \quad 2 \geq \alpha_2 \geq 1, \quad -1 \geq \alpha_3 \geq -2.\tag{13}$$

For $\alpha_1 = -1$ we have $\alpha_3 = -1$ and, according to (10), $R_3 = \text{const } R_1$; for $\alpha_1 = 1$ we have $\alpha_2 = 1$ and $R_2 = \text{const } R_1$. Both these limiting cases correspond to the flat model with axial symmetry ⁽²⁻⁴⁾. The values $|\alpha_1| < 1$ correspond to a completely anisotropic solution.

Denoting $\alpha_1 = 2 \sin \gamma$, where, according to (13), $-\pi/6 \leq \gamma \leq \pi/6$, we obtain expressions for the constants $\alpha_1, \alpha_2, \alpha_3$ in (10), satisfying (11) and (12), in the form ⁽¹⁾

$$\begin{aligned}\alpha_1 &= 2 \sin \gamma, & \alpha_2 &= 2 \sin(\gamma + 2\pi/3), & \alpha_3 &= 2 \sin(\gamma + 4\pi/3), \\ & & & -\pi/6 \leq \gamma \leq \pi/6.\end{aligned}\tag{14}$$

From (4) and (5), for the energy density it follows that

$$e = c^2 / [6\pi k (\tau - \tau_1)(\tau - \tau_2)].\tag{15}$$

At the instants of time τ_1 and τ_2 (we take $\tau_1 < \tau_2$), the quantity e becomes ∞ , and the solution has singularities. Under complete anisotropy ($|\alpha_1| < 1$), in

at the moment $\tau = \tau_1$, $R_1 = 0$, $R_2 = 0$, $R_3 = \infty$, and at the moment $\tau = \tau_2$, $R_1 = 0$, $R_2 = \infty$, $R_3 = 0$.

Owing to the arbitrariness in the exponents, the solution (10), generally speaking, cannot be analytically continued to both sides of the values $\tau = \tau_1$ and $\tau = \tau_2$. In this connection it should be noted that, along with (10), there are solutions differing from (10) by a change of sign at one or both of the differences $\tau - \tau_1$ and $\tau - \tau_2$. According to (15), for $\tau < \tau_1$ and for $\tau > \tau_2$, $e > 0$; in the interval $\tau_1 < \tau < \tau_2$, $e < 0$. We note that the solution (10) is not a symmetric function of τ (even when $\tau_2 = -\tau_1$). As $|\tau| \rightarrow \infty$, the solution (10) becomes isotropic: $R_1 \sim R_2 \sim R_3 \sim \tau^{2/3}$. For $\tau_1 = \tau_2$, (10) corresponds to the flat Friedmann model.

3. The ultrarelativistic equation of state $e = 3p$. From (6) we have $e = \text{const}/(-g)^{2/3}$. We shall consider physically real states with nonnegative values of $e = 3p$, so that

$$e = K/(-g)^{2/3}, \quad K = \text{const}, \quad K > 0. \quad (16)$$

Integrating (7) once, we obtain

$$[d(-g)^{1/2}/c d\tau]^2 = (24\pi kK/c^4)[(-g)^{1/3} + N], \quad N = \text{const}. \quad (17)$$

Values $N < 0$ in (17) do not lead to real R_1, R_2, R_3 ; this is again connected with the fact that for $N < 0$, in the sense of (17), the determinant $(-g)$ does not become 0, which is impossible in a synchronous system.

The value $N = 0$ in (17) corresponds to the isotropic flat Friedmann solution. Values $N > 0$ correspond to an anisotropic solution.

It is convenient to introduce the parameter λ according to

$$(-g)^{1/3} = L^2 \text{sh}^2 2\lambda, \quad L^2 \equiv N = \text{const}. \quad (18)$$

Equation (17) then takes the form

$$c d\tau/d\lambda = 6L^2 c^2 \text{sh}^2 2\lambda / (24\pi kK)^{1/2}.$$

Integrating (5), we obtain

$$c\tau/L_0 = \text{sh} 4\lambda - 4\lambda, \quad L_0 \equiv 3L^2 c^2 / 8(6\pi kK)^{1/2}. \quad (19)$$

The additive constant in (19) is chosen so that for $\lambda = 0$, $\tau = 0$ and $g = 0$.

Integrating (5), we obtain

$$\begin{aligned} R_1 &= L_1(\operatorname{sh} \lambda)^{1+\alpha_1}(\operatorname{ch} \lambda)^{1-\alpha_1}, & R_2 &= L_2(\operatorname{sh} \lambda)^{1+\alpha_2}(\operatorname{ch} \lambda)^{1-\alpha_2}, \\ R_3 &= L_3(\operatorname{sh} \lambda)^{1+\alpha_3}(\operatorname{ch} \lambda)^{1-\alpha_3}. \end{aligned} \quad (20)$$

L_1, L_2, L_3 are scale constants; $L_1 L_2 L_3 = 8L^3$.

The quantities $\alpha_1, \alpha_2, \alpha_3$ in (20) are constants for which the relations (11) and (12) again hold, arising as a consequence of (18) and after substitution of (20) into (4). For $\alpha_1, \alpha_2, \alpha_3$, formulas (13) and (14) are valid. The completely anisotropic solution again corresponds to values $|\alpha_1| < 1$.

As in the preceding case, the solution (20), generally speaking, is not analytically continued to both sides of the value $\tau = 0$ ($\lambda = 0$). In this case, for $\lambda < 0$, the solution is obtained from (20) by replacing $\operatorname{sh} \lambda$ with $-\operatorname{sh} \lambda$.

The “Hubble constants” for the solution (20) are equal to

$$h_{1,2,3} \equiv d \ln R_{1,2,3} / d\tau = [(\operatorname{ch} 2\lambda + \alpha_{1,2,3}) / \operatorname{sh}^3 2\lambda] (8\pi k K / 3c^2 L^4)^{1/2}. \quad (21)$$

At the moment $\tau = 0$ ($\lambda = 0$) the quantity e , according to (16), becomes ∞ , and the solution has a singularity.

In the completely anisotropic case ($|\alpha_1| < 1$), at the moment $\tau = 0$, $R_1 = R_2 = 0$, $R_3 = \infty$. Near $\tau = 0$ (for $\tau > 0$), $(R_1)^3 \sim \tau^{1+\alpha_1}$, $(R_2)^3 \sim$

$\sim \tau^{1+\alpha_2}$, $(R_3)^3 \sim \tau^{1+\alpha_3}$. For positive τ ($\lambda > 0$), according to (21) and (13), $h_1 > 0$ and $h_2 > 0$, so that as τ increases R_1 and R_2 increase monotonically; the quantity h_3 , however, changes sign at $\lambda = \lambda^*$ ($\operatorname{ch} 2\lambda^* = -\alpha_3$), so that R_3 first decreases and then increases.

In the completely anisotropic solution we have, for $\lambda > 0$, $h_3 < h_1 < h_2$.

In contrast to the case of complete anisotropy, under axial symmetry two types of solutions are possible, corresponding to the values $\alpha_1 = 1$ and $\alpha_1 = -1$. In the solution with $\alpha_1 = 1$ the character of the variation of R_1, R_2 , and R_3 is analogous to the preceding one; for $\tau > 0$ ($\lambda > 0$) here $h_1 = h_2 > h_3$. In the solution with $\alpha_1 = -1$ ($\alpha_2 = 2$, $\alpha_3 = -1$) the nature of the singularity at $\tau = 0$ is such that $R_1 = \operatorname{const}$, $\dot{R}_3 \neq 0$, $R_2 = 0$; for $\tau > 0$ ($\lambda > 0$) here $0 < h_1 = h_3 < h_2$, so that as τ increases, R_1, R_2 , and R_3 increase [4].*

As $|\tau| \rightarrow \infty$ the solution (20) becomes isotropized: $R_1 \sim R_2 \sim R_3 \sim |\tau|^{1/2}$.

4. The equation of state $e = p$. From (7) in this case we have

$$(-g)^{1/2} = \operatorname{const} \cdot \tau. \quad (22)$$

The origin of the time count has been chosen so that $g = 0$ at $\tau = 0$. From (5) we obtain

$$(R_1)^3 = \text{const } \tau^{1+\alpha_1}, \quad (R_2)^3 = \text{const } \tau^{1+\alpha_2}, \quad (R_3)^3 = \text{const } \tau^{1+\alpha_3}, \quad (23)$$

where the constants α_1, α_2 , and α_3 are connected in this case by one relation following from (22):

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \quad (24)$$

From (4) we obtain the expression for e :

$$e = c^2 [6 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)] / 144\pi k \tau^2. \quad (25)$$

We shall consider only solutions with nonnegative values of $e = p$. Then, according to (25), $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \leq 6$. Let

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 6q^2, \quad q = \text{const}, \quad 0 \leq q \leq 1. \quad (26)$$

Analogously to (13) and (14), in the present case we shall have

$$-q \leq \alpha_1 \leq q, \quad 2q \geq \alpha_2 \geq q, \quad -q \geq \alpha_3 \geq -2q, \quad (27)$$

$$\begin{aligned} \alpha_1 &= 2q \sin \gamma, \quad \alpha_2 = 2q \sin(\gamma + 2\pi/3), \quad \alpha_3 = 2q \sin(\gamma + 4\pi/3), \\ -\pi/6 &\leq \gamma \leq \pi/6, \quad 0 \leq q \leq 1. \end{aligned} \quad (28)$$

Equality signs correspond to axial symmetry, and strict inequality to complete anisotropy.

The moment $\tau = 0$ corresponds to a singularity ($e = \infty$). At $\tau = 0$, by (27), $R_1 = R_2 = 0$, while R_3 may be equal to ∞ , to 0, or to a finite nonzero value, depending on the values of the constants γ and q . For $q = 0$ we obtain the isotropic flat solution. For $q = 1$, according to (26) and (25), $e = p = 0$, and the solution (23) is the anisotropic solution for the vacuum [5]. The solution (23) for the equation of state $e = p$, in contrast to the solutions (10) and (20) for $p = 0$ and for $e = 3p$, does not become isotropized as $|\tau| \rightarrow \infty$ (this also applies to the anisotropic solution in the vacuum).

Scientific-Research Institute of Mechanics
Moscow State University
named after M. V. Lomonosov

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* For $p = 0$ the axially symmetric solutions (10) with $\alpha_1 = 1$ and with $\alpha_1 = -1$ reduce to one another. In these solutions, however, the singularity occurs at the values τ_1 and τ_2 , with different behavior of the solution in their vicinity.

Note: Figure translations are in progress. See original paper for figures.

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