

ON ORIENTED QUASICONFORMAL MAPPINGS IN SPACE

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1968

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Abstract

Full Text

UDC 517.53:512.9

MATHEMATICS

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ON ORIENTED QUASICONFORMAL MAPPINGS IN SPACE

(Presented by Academician M. A. Lavrent'ev on 30 I 1968)

1. Let E^n be n -dimensional Euclidean space, $|x|$ the length of the vector $x = (x_1, \dots, x_n) \in E^n$, and G a domain in E^n . By $C_0^\infty(G)$ we shall denote the class of infinitely differentiable finite functions with compact supports $\text{supp } u(x) \subset G$, and by $W_n^1(G)$ the class of vector functions $f(x) = [f_1(x), \dots, f_n(x)]$ having first-order partial derivatives, generalized in the sense of S. L. Sobolev, locally summable to the power n . Put

$$\lambda(x, f) = \left[\sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} \right)^2 \right]^{1/2}, \quad I(x, f) = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}.$$

A continuous vector function $f(x)$ is said to realize a q -quasiconformal mapping of a domain $G \subset E^n$ if $f(x) \in W_n^1(G)$ and there exists a constant q such that almost everywhere in the domain the inequality

$$\lambda^n(x, f) \leq n^{n/2} q^{n-1} |I(x, f)| \quad (1)$$

holds.

In the present note only oriented q -quasiconformal mappings are considered, i.e., mappings for which $I(x, f) \geq 0$ almost everywhere in the domain.

2. Below we shall need the following assertion, which we give here without proof.

Lemma 1. If a continuous vector function $f(x) \in W_n^1(G)$, $u(x) \in C_0^\infty(G)$, then

$$\left| \int_G u(x) I(x, f) dG \right| \leq C_1 \int_G |\nabla u(x)| |f_1(x)| \lambda^{n-1}(x, f) dG, \quad (2)$$

where C_1 is a constant depending only on n .

The following theorem is a simple consequence of Lemma 1.

Theorem 1. Let a vector function $f(x)$ realize an oriented q -quasiconformal mapping of a domain $G \subset E^n$.

Then for any compact set $F \Subset G$ the inequality

$$\|\lambda(x, f)\|_{L^n(F)} \leq C_2 \|f_1(x)\|_{L^n(G)} \quad (3)$$

holds, where C_2 depends only on n , q , and $\rho(F, \partial G)$.

This theorem makes it possible to sharpen somewhat the results of Yu. G. Reshetnyak ⁽¹⁾ and E. D. Callender ⁽²⁾ concerning the equicontinuity of q -quasiconformal mappings.

Theorem 2. Under the conditions of Theorem 1, for any compact set $F \Subset G$, for $x', x'' \in F$, one has

$$|f(x') - f(x'')| \leq C_3 |x' - x''|^\alpha, \quad (4)$$

where C_3 is a constant depending only on n , q , $\rho(F, \partial G)$, and $\|f_1(x)\|_{L^n(G)}$, and $0 < \alpha \leq 1$ depends on q and n .

This assertion follows directly from inequality (3) and Theorem 2 of paper ⁽¹⁾.

Theorem 3. Let $\{f(x)\}$ be a family of oriented $(1+\varepsilon)$ -quasiconformal mappings of the n -dimensional ($n > 2$) ball $G : |x| < 1$, normalized by the conditions: $f(0) = 0$, $f(1) = 1$, and having uniformly bounded norms $\|f_1(x)\|_{L^n(F)}$ on every compact set $F \subset G$.

Then there exists a universal function $\mu(\varepsilon, r) \geq 0$, defined for all $0 \leq r < 1$ and possessing the properties: $\mu(\varepsilon, r) \rightarrow 0$ as $\varepsilon \rightarrow 0$, for which

$$|f(x) - x| \leq \mu(\varepsilon, r). \quad (5)$$

The proof of this theorem is carried out analogously to ⁽³⁾, using the preceding theorem and Theorem 1 of paper ⁽⁴⁾.

For homeomorphic q -quasiconformal mappings this assertion was obtained by M. A. Lavrent'ev ⁽⁵⁾ (under some additional restrictions) and by Yu. G. Reshetnyak ⁽³⁾.

3. Let F be a compact set belonging to a domain $G \subset E^n$. Define its conformal capacity by

$$\text{cap}_G F = \inf \int_G |\nabla u(x)|^n dG, \quad (6)$$

where the infimum is taken over all functions $u(x) \in C_0^\infty(G)$ equal to 1 on F .

In proving the following two theorems we use

Lemma 2. If the vector function $f(x)$ realizes an oriented q -quasiconformal mapping of a domain $G \subset E^n$, then for any compact set $F \subset G$ one has

$$\|\lambda(x, f)\|_{L^n(F)} \leq C_4 \sup_G |f_1(x)| (\text{cap}_G F)^{1/n}, \quad (7)$$

where C_4 depends only on q and n .

The following two theorems contain results that overlap with already known earlier (see ⁽⁶⁾) assertions on the impossibility of mapping the whole space onto a part and on the removability of isolated singularities. Although our results are not generalizations of the known ones, they contain new information.

Theorem 4. If the vector function $f(x)$ realizes an oriented q -quasiconformal mapping of the entire space E^n and if

$$\max_{|x| \leq r} |f_1(x)| = o\left(\ln^{(n-1)/n} r\right) \quad (r \rightarrow \infty), \quad (8)$$

then

$$f(x) \equiv \text{const.}$$

Theorem 5. Let the vector function $f(x)$ realize an oriented q -quasiconformal mapping of the domain $G : 0 < |x| < 1$, and suppose that

$$\max_{r \leq |x| \leq 1/2} |f_1(x)| = o\left(\ln^{(n-1)/n} r\right) \quad (r \rightarrow 0), \quad (9)$$

$$\lim_{x \rightarrow 0} |f(x)| \neq \infty. \quad (10)$$

Then $f(x)$ is continuous at the origin.

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Received
25 I 1968

References

- ¹ Yu. G. Reshetnyak, *Sibirsk. matem. zhurn.*, **7**, No. 5 (1966).
- ² E. D. Kalleder, *Pacific J. Math.*, **10**, No. 2 (1960).
- ³ Yu. G. Reshetnyak, In: *Some Problems of Mathematics and Mechanics*, Novosibirsk, 1961.

⁴ Yu. G. Reshetnyak, *Sibirsk. matem. zhurn.*, **8**, 4 (1967).

⁵ M. A. Lavrent'ev, *DAN*, **95**, No. 5 (1954).

⁶ Yu. G. Reshetnyak, B. V. Shabat, *Proceedings of the IV Conference on Function Theory*, 2, L., 1964.

Note: Figure translations are in progress. See original paper for figures.

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