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Abstract

Full Text

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MATHEMATICS

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ON A NONLINEAR SYSTEM OF EQUATIONS OF MIXED TYPE

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Problem T. In the domain D , find a regular solution of the system

$$E(z_i) = y \partial^2 z_i / \partial x^2 + \partial^2 z_i / \partial y^2 = f_i(x, y, z_1, \dots, z_n), \quad (1)$$

taking the prescribed values:

$$z_i|_{AC} = \psi_i(t), \quad z_i|_{\Gamma} = \varphi_i(S), \quad i = 1, 2, \dots, n, \quad (2)$$

where the φ_i are continuously differentiable and the ψ_i are twice continuously differentiable functions.

The domain D is bounded by a Jordan curve Γ in the half-plane $y > 0$, with endpoints at the points $A(0, 0)$, $B(1, 0)$, and by arcs of the characteristics AC and BC of the equation $E(z) = 0$, if $y < 0$. The functions f_i are assumed to be continuous and to have second derivatives in the domain $x, y \in \overline{D}$, $|z_i| \leq c$, and moreover $\partial f_i / \partial z > 0$.

We first establish some properties of solutions of the linear system

$$yz_{xx} + z_{yy} - cz = 0, \quad (3)$$

where $z = (z_1, z_2, \dots, z_n)$ is a vector, and $c(x, y)$ is a positive definite $n \times n$ matrix $c = \|c_{ik}\|$, whose components satisfy the conditions

$$(n-1)(c_{ik} + c_{ki}) \leq 2\sqrt{c_{ii}c_{kk}}, \quad i \neq k. \quad (4)$$

Let

$$R(x, y) = \left(\sum_{k=1}^n z_k \right)^{1/2};$$

D_1, D_2 are the parts of the domain D in the half-planes respectively $y > 0$, $y < 0$.

Lemma 1. The function $R(x, y)$ cannot attain a positive maximum at an interior point of the domain D_1 (see ⁽¹⁾).

Supposing that $R(x, y)$ attains a positive maximum at an interior point P of the domain D_1 , we obtain

$$E(R(P)) \leq 0. \quad (5)$$

On the other hand, by virtue of (3),

$$E(R) = \frac{1}{R} [y(z_x)^2 + (z_y)^2 + zcz - y(R_x)^2 - (R_y)^2]. \quad (6)$$

Consequently, at the point P ,

$$E(R) > 0. \quad (7)$$

The contradiction between inequalities (5) and (7) proves the lemma.

Lemma 2. If at some point $x = x_0$ of the segment $(0, 1)$ of the axis $y = 0$, $R(x, y)$ assumes its greatest positive value, and if the values of $R(x, y)$ on Γ are less than $R(x_0, 0)$, then

$$\overline{R}(x_0) = \lim_{y \rightarrow 0} \partial R(x_0, y) / \partial y < 0 \quad (8)$$

provided that this limit exists.

For degenerating elliptic equations this proposition is given in work ⁽²⁾.

Obviously, $\overline{R}(x_0) > 0$ cannot occur. Suppose that $\overline{R}(x_0) = 0$. Let $\mu > 0$, let d be the diameter of the domain D_1 , and let $R(x_0, 0) = 1$. Consider the function

$$u = \varepsilon R / (e^{\mu d} - \varepsilon e^{\mu y}). \quad (9)$$

It is easy to see that the function $u(x, y)$ must have a positive maximum at an interior point P of the domain D_1 , and, by virtue of (9),

$$E(R(P)) = \frac{1}{\varepsilon} \left[E(u) - \frac{2\varepsilon e^{\mu y}}{e^{\mu d} - \varepsilon e^{\mu y}} u_y - \frac{\varepsilon \mu e^{\mu y}}{e^{\mu d} - \varepsilon e^{\mu y}} u \right] < 0. \quad (10)$$

On the other hand, at the point P , taking (9) into account, we obtain $R_x = 0$, $R_y = -\mu e^{\mu y} u$, and, by virtue of (3) and (4), from (6) we obtain the inequality $E(R(P)) > 0$, contradicting (10), which proves the lemma.

Lemma 3. In the domain D_2 there exists a unique continuously differentiable solution of system (3), assuming on the boundary the prescribed values

$$z_i|_{AC} = \psi_i(t), \quad z_i|_{AB} = \tau_i(x). \quad (11)$$

Let $z_i^{(0)}$ be continuous solutions of the equation $E(z_i) = 0$ in the domain $\overline{D}_2$, assuming on the boundary the values (11). Such solutions are known (2).

Let

$$u_i = z_i - z_i^{(0)}. \quad (12)$$

Then the functions u_i satisfy the equation

$$E(u_i) - \sum_{k=1}^n c_{ik}(u_k + z_k^{(0)}) = 0 \quad (13)$$

and the homogeneous boundary conditions (11). Setting $u_i^{(0)} = 0$, and

$$u_i^{(m)} = \lambda \int_0^\xi d\xi' \int_{\xi'}^\eta \frac{V(\xi, \eta; \xi', \eta')}{(\eta' - \xi')^{2/3}} \sum_{k=1}^n c_{ik}(u_k^{(m-1)} + z_k^{(0)}) d\eta', \quad (14)$$

we find that

$$|u_i^{(m+1)} - u_i^{(m)}| \leq 2cM^{m+1}(\eta - \xi)^{1/3}\eta^{1/3+(m+1)/3} \frac{\xi^m}{m!}, \quad (15)$$

where $c = \max\{|z_k^{(0)}|\}$, $M = 6\lambda Nn$, $N = \max|c_{ik}|$ for all $i, k = 1, 2, \dots, n$.

From the estimates (15) follows the uniform convergence of the sequence $u_i^{(m)}$ to the solution of equation (13).

Lemma 4. If the function $R(\xi, \eta)$ is equal to zero on AC and assumes its greatest positive value at some point $(x_0, 0)$ on AB , then there exists in D_2 a neighborhood of this point in which

$$R(\xi, \eta) < R(x_0, 0). \quad (16)$$

From (12) we have

$$z_i = k \int_0^\xi \tau_i(t) \frac{(\eta - \xi)^{2/3} dt}{(\eta - t)^{5/6}(\xi - t)^{5/6}} + u_i(\xi, \eta), \quad (17)$$

where $u_i = \lim_{m \rightarrow \infty} u_i^{(m)}$.

Assuming that z_i on AB attains its greatest positive value at the point $(x_0, 0)$, and taking into account the estimates (15), we obtain

$$z_i \leq \tau_i(x_0) \left\{ 1 - \left(\frac{\eta - \xi}{\eta} \right)^{2/3} [3/2 k - 2cM\overline{M}\eta^{1/3}] \right\}, \quad (18)$$

where $\overline{M} = \max e^\xi \eta^{1/3}$.

It follows from inequality (18) that if $\eta < (3k/4cM\overline{M})^{3/4}$, then $z_i(\xi, \eta) < \tau_i(x_0)$. Since $z_i(\xi, \eta)$ in some neighborhood of the point $(x_0, 0)$, by virtue of continuity, as well as $\tau(x_0)$, is positive, we obtain inequality (16) in an obvious way.

Lemma 5. If the functions $z_i(x, y)$ satisfy equation (3) in the domain D , and are equal to zero on the characteristic AC , then the norm $R(x, y)$ on the segment AB cannot assume a greatest positive value.

By virtue of the first lemma, $R(x, y)$ in D_1 cannot have a positive maximum. Assuming that this maximum is attained on AB , by virtue of (8) and (16) we arrive at a contradiction.

Theorem 1. The solution of problem T for system (3) is unique in D . The proof follows easily from Lemmas 1, 3, and 5.

Theorem 2. If

$$(n-1) \left(\frac{\partial f_i}{\partial z_k} + \frac{\partial f_k}{\partial z_i} \right) \leq 2 \left(\frac{\partial f_i}{\partial z_i} \frac{\partial f_k}{\partial z_k} \right)^{1/2},$$

then in the domain D there exists a solution of problem T for equation (1).

Supposing that problem T for system (1) has two solutions z_i and w_i , we obtain that the difference $u_i = z_i - w_i$ satisfies the system

$$E(u_i) = \sum_{k=1}^n \frac{\partial f_i}{\partial u_k} u_k,$$

which possesses all the properties of system (3) and, consequently, by virtue of the first theorem has the unique zero solution.

We preface the proof of the existence of a solution by two lemmas.

Lemma 6. In the domain D_1 , for system (1) there exists a unique twice continuously differentiable solution which assumes the prescribed values on the boundary

$$z_i|_{\Gamma} = \varphi_i(s), \quad dz_i/dy = \nu_i(x). \quad (19)$$

The uniqueness of the solution follows from Lemmas 1 and 2. Let $z_i^{(0)}$ be the solution of the equation $E(z_i) = 0$, satisfying conditions (19).

Putting

$$v_i = z_i - z_i^{(0)}, \quad (20)$$

we find that v_i satisfy the equation $E(v_i) = f_i(x, y, v_1 + z_1^{(0)}, \dots, \dots, v_n + z_n^{(0)})$ and the homogeneous conditions (19). Replacing this equation by an integral one and solving it by the method of successive approximations, where $v_i^{(0)} = 0$, and

$$v_i^{(m+1)} = \iint_{D_1} f_i(\xi, \eta, v_1^{(m)} + z_1^{(0)}, \dots, v_n^{(m)} + z_n^{(0)}) G(x, y; \xi, \eta) d\xi d\eta,$$

we obtain that the functions $v_i = \lim_{m \rightarrow \infty} v_i^{(m)}$ solve the posed problem.

For v_i the estimates

$$|dv_i(x, 0)/dx| \leq c, \quad c = \text{const.} \quad (21)$$

are valid.

Lemma 7. In the domain D_2 there exists a unique continuously differentiable solution of system (1), which assumes on the boundary the values

$$z_i|_{AC} = \psi_i(t), \quad z_i|_{AB} = \nu_i(x) \quad (22)$$

and has the form

$$z_i = u_i + z_i^{(0)}, \quad u_i = \lim_{m \rightarrow \infty} u_i^{(m)}, \quad (23)$$

where

$$z_i^{(0)}(\xi, \eta) = k \int_0^\xi \nu_i(t) (\xi - t)^{-1/6} (\eta - t)^{-1/6} dt + \int_0^b \left(\psi_i' + \frac{\psi_i}{6t} \right) V(\xi, \eta; 0, t) dt, \quad (24)$$

$$u_i^{(m+1)}(\xi, \eta) = \lambda \int_0^\xi d\xi' \int_\xi^\eta \frac{V(\xi, \eta; \xi', \eta')}{(\eta' - \xi')^{2/3}} f_i(\xi, \eta, u_1^{(m)} + z_1^{(0)}, \dots, u_n^{(m)} + z_n^{(0)}) d\eta'.$$

For $u_i(\xi, \eta)$, when $\eta = \xi = x$, we have the estimates

$$|du_i(x, x)/dx| \leq c, \quad c = \text{const.} \quad (25)$$

Now from equations (20) and (23), along the segment AB , one can form a system of equations with respect to $\tau_i(x)$ and $\nu_i(x)$.

Eliminating $\tau_i(x)$ from this system, we reduce the solution of problem T to the solution of a singular integral equation with respect to $\nu_i(x)$ (see (2)).

By virtue of the estimates (21) and (25), this singular equation is reduced in the usual way to a Fredholm equation, whose solvability follows from the uniqueness of the solution proved above.

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