

ON ALGORITHMIC PROBLEMS

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1968

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Abstract

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MATHEMATICS

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ON ALGORITHMIC PROBLEMS

FOR COMMUTATIVE SEMIGROUPS

(Presented by Academician A. I. Mal' tsev on IV 8, 1967)

1. Algorithms for solving various problems in the theory of commutative semigroups were proposed by A. I. Mal' tsev ⁽¹⁾, Tseitlin, Biryukov ⁽²⁻⁵⁾, Emelichev ⁽⁶⁾, and Khalezov ⁽⁷⁾. In papers ^(8,9) the author described a universal algorithm which solves any problem, provided only that it is written in the language of first-order predicate calculus of a definite signature. For a more precise formulation of these results, with each commutative semigroup $L(\mathfrak{A}, \mathfrak{B})$, given by a finite set of generators $\mathfrak{A}$ and a set of defining relations $\mathfrak{B}$, associate the signature $\sigma(\mathfrak{A}) = \langle a, G_a \mid a \in \mathfrak{A} \rangle$, where G_a for $a \in \mathfrak{A}$ is a unary predicate symbol. We shall say that an algorithm **solves problem A** if, from $\mathfrak{A}, \mathfrak{B}$ and from an arbitrary closed formula Φ of first-order predicate calculus of signature $\sigma(\mathfrak{A})$, this algorithm determines whether the formula Φ is true in the semigroup $L(\mathfrak{A}, \mathfrak{B})$ in the case when the predicate $G_a(x)$ for each $a \in \mathfrak{A}$ is interpreted in $L(\mathfrak{A}, \mathfrak{B})$ as “ x is equal to some natural multiple of a .” The principal content of papers ^(8,9) is the description of an algorithm solving problem A.

The algorithm described consisted in successively obtaining consequences from a certain system of axioms until either the formula Φ or the formula $\neg\Phi$ was derived. A certain shortcoming of papers ^(8,9) was the use of model-theoretic considerations in proving completeness of the system of axioms used. This distinguished the description of the universal algorithm from constructive descriptions of algorithms for solving individual problems in the cited papers.

In the present note another algorithm is described, also solving problem A. The description given here is constructive and simpler. The following result of this note may also be of independent interest. If by the elements of a commutative semigroup with n generators one understands equivalence classes defined on n -tuples of natural numbers, then the corresponding equivalence relation is always definable on the natural numbers by some formula of first-order predicate calculus containing no nonlogical symbols other than the symbol of the addition operation.

2. In what follows, without explanation, the terminology adopted in item 1 of § 1 of the survey ⁽¹⁰⁾ is used. Let an algebraic system $\mathfrak{M}$ of signature σ have a decidable theory. Let $\mathfrak{G}(x_1, \dots, x_k, y_1, \dots, y_k)$ be a formula of signature σ , containing no free variables other than $x_1, \dots, x_k, y_1, \dots, y_k$. Let the relation $\sim$, defined in the system $\mathfrak{M}^k$ by the condition: for $(x_1, \dots, x_k), (y_1, \dots, y_k)$ from $\mathfrak{M}^k$ if and only if $(x_1, \dots, x_k) \sim (y_1, \dots, y_k)$, when $\mathfrak{G}(x_1, \dots, x_k, y_1, \dots, y_k)$ is true in $\mathfrak{M}$, be a congruence relation in $\mathfrak{M}^k$. Then the factor system $\mathfrak{M}^k / \sim$ also has a decidable theory. If the relation $\sim$ and the formula $\mathfrak{G}(x_1, \dots, x_k, y_1, y_2, \dots, y_k)$ are connected by the condition considered above, we shall say that the relation $\sim$ is elementary in $\mathfrak{M}$ and that the formula $\mathfrak{G}(x_1, \dots, x_k, y_1, \dots, y_k)$ defines the relation $\sim$.

It is known that the system $\mathfrak{N} = \langle N, + \rangle$, where $N = \{0, 1, 2, \dots\}$ is the set of natural numbers and $+$ is the usual operation of addition of natural numbers, numbers, has a decidable theory. Therefore every factor system of the system $\mathfrak{N}^k$ with respect to an elementary equivalence relation also has a decidable theory.

Each semigroup $L(\mathfrak{A}, \mathfrak{L})$ can be regarded as a factor semigroup of the semigroup $\mathfrak{N}^k$, where k is the number of elements of $\mathfrak{A}$. To solve problem A it is therefore enough to note that this factor semigroup is obtained by factoring $\mathfrak{N}^k$ with respect to an elementary equivalence relation and that the formula defining this equivalence relation is effectively constructed from $\mathfrak{A}$ and $\mathfrak{B}$.

In the case where $L(\mathfrak{A}, \mathfrak{L})$ is a semigroup with cancellation, this is observed trivially. Indeed, let $\mathfrak{A} = \{a_1, \dots, a_k\}$, $\mathfrak{L} = \{(c_i, d_i) \mid i = 1, \dots, q\}$, where

$$c_i = \sum_{j=1}^k \alpha_j^{(i)} a_j; \quad d_i = \sum_{j=1}^k \beta_j^{(i)} a_j; \quad \alpha_j^{(i)}, \beta_j^{(i)} \in N.$$

In this case $(x_1, \dots, x_k) \sim (y_1, \dots, y_k)$ is equivalent to the condition

$$(\exists t_1) \dots (\exists t_q) (\exists t'_1) \dots (\exists t'_q) \left(\bigwedge_{i=1}^k \left[x_i + \sum_{j=1}^q t_j (\alpha_i^{(j)} - \beta_i^{(j)}) = y_i + \sum_{j=1}^q t'_j (\alpha_i^{(j)} - \beta_i^{(j)}) \right] \right).$$

This condition can obviously be represented in the form of the required formula.

3. In this paragraph we recall some definitions and results from § 1 of [9]. By $L(\mathfrak{A})$ we denote the semigroup, free in the class of commutative semigroups with zero, with set $\mathfrak{A}$ of free generators. We think of the set $L(\mathfrak{A})$ as the set of all possible linear forms in the letters $a_1, \dots, a_k$, where $\mathfrak{A} = \{a_1, \dots, a_k\}$, with natural coefficients. A semigroup $L(\mathfrak{A}, \mathfrak{L})$, given in the class of commutative semigroups with zero by finite sets of generators $\mathfrak{A}$ and defining relations $\mathfrak{B}$, is regarded by us as a factor semigroup of the semigroup $L(\mathfrak{A})$. For $x \in L(\mathfrak{A})$, by $\bar{x}$ we denote the image of x

under the canonical mapping $L(\mathfrak{A}) \rightarrow L(\mathfrak{A}, \mathfrak{L})$. By $M(\mathfrak{A}, \mathfrak{L})$ we denote the semigroup, given in the class of commutative semigroups with cancellation and with zero, by finite sets of generators $\mathfrak{A}$ and defining relations $\mathfrak{B}$. We also regard the semigroup $M(\mathfrak{A}, \mathfrak{L})$ as a factor semigroup of the semigroup $L(\mathfrak{A})$. For $x \in L(\mathfrak{A})$, by $[x]$ we denote the image of x under the canonical mapping $L(\mathfrak{A}) \rightarrow M(\mathfrak{A}, \mathfrak{L})$.

For $a \in \mathfrak{A}$ and $f \in L(\mathfrak{A})$, by $(f)_a$ we denote the coefficient of the letter a in the form f . For $f, g \in L(\mathfrak{A})$ we write $f \geq g$ if $(f)_a \geq (g)_a$ for all $a \in \mathfrak{A}$. We write $\mathfrak{A}_1 \subset \mathfrak{A}$ if $\mathfrak{A}_1 \neq \mathfrak{A}$ and $\mathfrak{A}_1$ is a subset of the set $\mathfrak{A}$. The symbol $\emptyset$ denotes the empty set. By $L(\emptyset)$ we denote the zero semigroup. If $\mathfrak{A}_1 \subset \mathfrak{A}$, the semigroup $L(\mathfrak{A}_1)$ is regarded as a subsemigroup of the semigroup $L(\mathfrak{A})$. We assume that $\mathfrak{A} \subseteq L(\mathfrak{A})$. For $\mathfrak{A}_1 \subset \mathfrak{A}$ and $f \in L(\mathfrak{A})$, by $\mathfrak{A}_1(f)$ we denote such a form from $L(\mathfrak{A} \setminus \mathfrak{A}_1)$ that $(\mathfrak{A}_1(f))_a = (f)_a$ for all $a \in \mathfrak{A} \setminus \mathfrak{A}_1$. Let $q(\mathfrak{L})$ be the number of relations in $\mathfrak{B}$ and $\lambda(\mathfrak{L})$ the greatest coefficient in these relations. By $h(\mathfrak{A}, \mathfrak{L})$ we denote the form

$$\sum_{a \in \mathfrak{A}} \lambda(\mathfrak{L})(q(\mathfrak{L}) + 1)a$$

from $L(\mathfrak{A})$.

Let $\mathfrak{A}_1 \subset \mathfrak{A}$. By $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{L})$ we denote the set of all $x \in L(\mathfrak{A} \setminus \mathfrak{A}_1)$ such that the inequality $\overline{x + y} \neq \overline{h(\mathfrak{A}, \mathfrak{L}) + z}$ holds in $L(\mathfrak{A}, \mathfrak{L})$ for all $y \in L(\mathfrak{A}_1)$, $z \in L(\mathfrak{A})$. By $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{L})$ we denote the set of elements of $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{L})$ that are maximal in the sense of the relation $\geq$. By $\mathfrak{M}_1(\mathfrak{A}, \mathfrak{L})$ we denote the set

$$\{x \in L(\mathfrak{A} \setminus \mathfrak{A}_1) \mid (\exists y)(y \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{L}) \ \& \ y \geq x)\}.$$

The sets $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{L})$ and $\mathfrak{M}_1(\mathfrak{A}, \mathfrak{L})$ are finite.

We divide the set $\mathfrak{M}_1(\mathfrak{A}, \mathfrak{L})$ into reducibility classes, assigning $x, y \in \mathfrak{M}_1(\mathfrak{A}, \mathfrak{L})$ to one class if there exist $z, u \in L(\mathfrak{A}_1)$ such that $\overline{x + z} = \overline{y + u}$ in $L(\mathfrak{A}, \mathfrak{L})$. By $i(\mathfrak{A}_1, \mathfrak{L})$ we denote the number of reducibility classes, and by

$$\mathfrak{M}_{2,1}(\mathfrak{A}, \mathfrak{L}), \dots, \mathfrak{M}_{2,i(\mathfrak{A}_1, \mathfrak{L})}(\mathfrak{A}, \mathfrak{L})$$

all these distinct classes.

For $x, y \in \mathfrak{M}_{2,j}(\mathfrak{A}, \mathfrak{L})$, $j \in \{1, \dots, i(\mathfrak{A}_1, \mathfrak{L})\}$, $\mathfrak{A}_3 \subset \mathfrak{A}_1$, by $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{L})$ we denote the set of all such $z \in L(\mathfrak{A}_1 \setminus \mathfrak{A}_3)$ that for all $u \in L(\mathfrak{A}_3)$, $v \in L(\mathfrak{A}_1)$ in $L(\mathfrak{A}, \mathfrak{L})$ we have the inequality $\overline{x + z + u} \neq \overline{y + v}$. By $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{L})$ we denote the set of maximal elements of $\mathfrak{N}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{L})$. The set $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{L})$ is finite. Recall that:

- (1) the sets $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{L})$, $\mathfrak{M}_1(\mathfrak{A}, \mathfrak{L})$, $\mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{L})$ are effectively constructed from $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, \mathfrak{A}_3, x, y$ ⁽⁹⁾;

- (2) the set $\mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{L})$ is effectively, from $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1$, divided into reducibility classes ⁽⁹⁾;
- (3) there exists an effective procedure which, from $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, x$ such that $\mathfrak{A}_1 \subset \mathfrak{A}$ and $x \in L(\mathfrak{A} \setminus \mathfrak{A}_1)$, constructs such a finite set $\mathfrak{L}(\mathfrak{A}_1, x)$ of defining relations for $\mathfrak{A}_1$ that for $u, v \in L(\mathfrak{A}_1)$ then and only then $\bar{u} = \bar{v}$ in $L(\mathfrak{A}_1, \mathfrak{L}(\mathfrak{A}_1, x))$, when $\overline{x+u} = \overline{x+v}$ in $L(\mathfrak{A}, \mathfrak{L})$ ⁽⁹⁾;
- (4) for $x, y \in L(\mathfrak{A})$ then and only then $\overline{x+h(\mathfrak{A}, \mathfrak{L})} = \overline{y+h(\mathfrak{A}, \mathfrak{L})}$ in $L(\mathfrak{A}, \mathfrak{L})$, when $[x] = [y]$ in $M(\mathfrak{A}, \mathfrak{L})$ (Zeitlin's lemma, (7));
- (5) for $x \in L(\mathfrak{A})$ either there exists $z \in L(\mathfrak{A})$ such that $\bar{x} = \overline{h(\mathfrak{A}, \mathfrak{L}) + z}$ in $L(\mathfrak{A}, \mathfrak{L})$, or there exists $\mathfrak{A}_1 \subset \mathfrak{A}$ such that $\mathfrak{A}_1(x) \in \mathfrak{M}_1(\mathfrak{A}_1, \mathfrak{L})$ ⁽⁹⁾;
- (6) for $\mathfrak{A}_1 \subset \mathfrak{A}$, $x, y \in \mathfrak{M}_{2,j}(\mathfrak{A}_1, \mathfrak{L})$, $j \in \{1, \dots, i(\mathfrak{A}_1, \mathfrak{L})\}$, $u \in L(\mathfrak{A}_1)$, either there exists $v \in L(\mathfrak{A}_1)$ such that $\bar{x+u} = \bar{y+v}$ in $L(\mathfrak{A}, \mathfrak{L})$, or there exist $\mathfrak{A}_3 \subset \mathfrak{A}_1$ and $v \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, x, y, \mathfrak{L})$ such that $\mathfrak{A}_3(u) \leq v$ ⁽⁹⁾.

4. Theorem. Let $\mathfrak{A} = \{a_1, \dots, a_k\}$. The equivalence relation in $\mathfrak{N}^k$, defined by the condition

$$(\alpha_1, \dots, \alpha_k) \sim (\beta_1, \dots, \beta_k)$$

if and only if

$$\sum_{i=1}^k \alpha_i \bar{a}_i = \sum_{i=1}^k \beta_i \bar{a}_i$$

in $L(\mathfrak{A}, \mathfrak{L})$, is elementary in $\mathfrak{N}$. There exists an effective procedure which, from $\mathfrak{A}$ and $\mathfrak{B}$, constructs a formula

$$\mathfrak{C}(\mathfrak{A}, \mathfrak{L}; \alpha_a, \beta_a \mid a \in \mathfrak{A})$$

defining the relation $\sim$.

Proof is carried out by induction on the number of elements of $\mathfrak{A}$. For a one-element $\mathfrak{A}$ it is obvious. Suppose that we can construct

$$\mathfrak{C}(\mathfrak{A}, \mathfrak{B}; \alpha_a, \beta_a \mid a \in \mathfrak{A})$$

in the case when $\mathfrak{A}$ contains fewer than k elements. Let $\mathfrak{A}$ contain k elements. Let

$$\mathfrak{C}^*(\mathfrak{A}, \mathfrak{L}; \alpha_a, \beta_a \mid a \in \mathfrak{A})$$

denote a formula which defines such a congruence relation $\sim_1$ in $\mathfrak{N}^k$ that the semigroups $M(\mathfrak{A}, \mathfrak{L})$ and $\mathfrak{N}^k / \sim_1$ are isomorphic.

For $\mathfrak{A}_1 \subset \mathfrak{A}$, $j \in \{1, \dots, i(\mathfrak{A}_1, \mathfrak{L})\}$, $u, v \in \mathfrak{M}_{2,j}(\mathfrak{A}, \mathfrak{L})$, by $\Phi(\mathfrak{A}_1, u, v)$ we denote the set of all such functions φ which to each $\mathfrak{A}_3 \subset \mathfrak{A}_1$ and each $e \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, u, v, \mathfrak{L})$ assign an element $\varphi(\mathfrak{A}_3, e) \in \mathfrak{A}_1 \setminus \mathfrak{A}_3$. For $\varphi \in \Phi(\mathfrak{A}_1, u, v)$ consider the set

$$\mathfrak{A}(\varphi) = \{\varphi(\mathfrak{A}_3, e) \mid \mathfrak{A}_3 \subset \mathfrak{A}_1, e \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, u, v, \mathfrak{L})\}.$$

For $a \in \mathfrak{A}(\varphi)$ by μ_a we denote the natural number

$$\max\{(e)_a \mid \mathfrak{A}_3 \subset \mathfrak{A}_1, e \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{A}_3, u, v, \mathfrak{L}), \varphi(\mathfrak{A}_3, e) = a\}.$$

By $F_\varphi(u, v, \mathfrak{A}_1)$ we denote the conjunction of all formulas

$$\alpha_a = (u)_a, \quad \beta_a = (v)_a, \quad \alpha_c = \mu_c + z_c + 1$$

for all possible $a \in \mathfrak{A} \setminus \mathfrak{A}_1$, $c \in \mathfrak{A}(\varphi)$, where $z_c, \alpha_c, \alpha_a, \beta_a$ are symbols of object variables.

Let

$$d = \sum_{a \in \mathfrak{A} \setminus \mathfrak{A}_1} \alpha_a a + \sum_{a \in \mathfrak{A}(\varphi)} (\mu_a + 1)a.$$

From (6) it follows that $\bar{d} = \bar{v} + \bar{d}_1$ in $L(\mathfrak{A}, \mathfrak{L})$ for some $d_1 \in L(\mathfrak{A}_1)$. Now in $\mathfrak{C}(\mathfrak{A}, \mathfrak{L}(\mathfrak{A}_1, v); \alpha_a, \beta_a \mid a \in \mathfrak{A}_1)$, in place of α_a substitute $(d_1)_a + z_a$ if $a \in \mathfrak{A}(\varphi)$, and substitute $\alpha_a + (d_1)_a$ if $a \notin \mathfrak{A}(\varphi)$. We obtain the formula $G_\varphi(u, v, \mathfrak{A}_1)$. By $D_\varphi(u, v, \mathfrak{A}_1)$ we denote the formula that is obtained if, before the formula $F_\varphi(u, v, \mathfrak{A}_1)$ & $G_\varphi(u, v, \mathfrak{A}_1)$, one prefixes existential quantifiers over z_c for all $c \in \mathfrak{A}(\varphi)$. By $D(\mathfrak{A}_1, u, v)$ we denote the disjunction of the formulas $D_\varphi(u, v, \mathfrak{A}_1)$ for all $\varphi \in \Phi(\mathfrak{A}_1, u, v)$.

By Ψ we denote the set of all such functions ψ that to each $\mathfrak{A}_1 \subset \mathfrak{A}$ and to each $c \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{L})$ assign an element $\psi(\mathfrak{A}_1, c) \in \mathfrak{A} \setminus \mathfrak{A}_1$. For $\psi \in \Psi$ consider the set $\mathfrak{A}_\psi = \{\psi(\mathfrak{A}_1, c) \mid \mathfrak{A}_1 \subset \mathfrak{A}, c \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{L})\}$. For $a \in \mathfrak{A}_\psi$, by ν_a we denote the natural number

$$\max\{(c)_a \mid \mathfrak{A}_1 \subset \mathfrak{A}, c \in \mathfrak{M}(\mathfrak{A}_1, \mathfrak{L}), \psi(\mathfrak{A}_1, c) = a\}.$$

By $H_\psi(\alpha_c, z_c \mid c \in \mathfrak{A}_\psi)$ we denote the conjunction of all formulas $\alpha_c = \nu_c + z_c + 1$ for all possible $c \in \mathfrak{A}_\psi$, where α_c, z_c are symbols for object variables.

Let

$$f_\psi = \sum_{a \in \mathfrak{A}_\psi} (\nu_a + 1)a.$$

From (5) it follows that $\bar{f}_\psi = \overline{h(\mathfrak{A}, \mathfrak{L})} + g_\psi$ for some $g_\psi \in L(\mathfrak{A})$. By $B(\psi, \tau)$ we denote the formula obtained if, in $\mathfrak{C}^*(\mathfrak{A}, \mathfrak{L}; \alpha_a, \beta_a \mid a \in \mathfrak{A})$, one substitutes $z_c + (g_\psi)_c$ for α_c , for $c \in \mathfrak{A}_\psi$; substitutes $\alpha_a + (g_\psi)_a$ for α_a , for all $a \in \mathfrak{A} \setminus \mathfrak{A}_\psi$; substitutes $y_d + (g_\tau)_d$ for β_d , for all $d \in \mathfrak{A}_\tau$; and substitutes $\beta_a + (g_\tau)_a$ for β_a , for all $a \in \mathfrak{A} \setminus \mathfrak{A}_\tau$. By $E(\psi, \tau)$ we denote the formula obtained if, before the conjunction

$$H_\psi(\alpha_c, z_c \mid c \in \mathfrak{A}_\psi) \ \& \ H_\tau(\beta_d, y_d \mid d \in \mathfrak{A}_\tau) \ \& \ B(\psi, \tau)$$

one prefixes existential quantifiers over all z_c and y_d , for $c \in \mathfrak{A}_\psi$ and $d \in \mathfrak{A}_\tau$.

From (4), (5), (6) it follows that as $\mathfrak{C}(\mathfrak{A}, \mathfrak{L}; \alpha_a, \beta_a \mid a \in \mathfrak{A})$ one may take the disjunction of all formulas $E(\psi, \tau)$, $D(\mathfrak{A}_1, u, v)$ for all possible $\psi, \tau \in \Psi$, all possible $\mathfrak{A}_1 \subset \mathfrak{A}$, all possible $j \in \{1, \dots, i(\mathfrak{A}_1, \mathfrak{L})\}$, and all possible $u, v \in \mathfrak{M}_{2,j}(\mathfrak{A}_1, \mathfrak{L})$.

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Received
31 III 1967.

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