

ON BOUNDED SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract

Full Text

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MATHEMATICS

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ON BOUNDED SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER

(Presented by Academician L. S. Pontryagin on July 5, 1967)

In the present work we give some results concerning the existence, uniqueness, and behavior as $x \rightarrow \infty$ of solutions, bounded for all x , of systems of the form

$$y'' = F(x, y, y'), \quad (*)$$

where y and $F(x, y, z)$ are n -dimensional vectors. In the case of a single equation, analogous results were obtained in papers ⁽¹⁻³⁾; in the case of the system $y'' = \psi(x, y)$, in paper ⁽⁴⁾. Periodic solutions of system $(*)$ were considered in paper ⁽⁵⁾. Related questions for first-order systems were studied in the monograph ⁽⁶⁾.

Theorem 1. Let the equation

$$y'' = f(x, y, y'), \quad (1)$$

be given, where the derivatives $f_y(x, y, z)$, $f_z(x, y, z)$ are bounded in every bounded region of the space x, y, z , and $f_y(x, y, z) > 0$ for all values of z and such values of x, y to which the whole xoy -plane corresponds, except for a set of measure 0.

Then, if equation (1) has a solution bounded for all values of x , it is unique.

The proof is based on the maximum principle, which can be formulated in the form of the following lemma:

Lemma. Let the equation

$$y'' = f(x)y + \varphi(x)y' + \psi(x), \quad (1')$$

be given, where the functions $f(x)$, $\varphi(x)$, $\psi(x)$ are bounded in some interval (a, b) , and almost everywhere in (a, b) the inequalities $f(x) > 0$, $\psi(x) \geq 0$ (≤ 0)

hold. Then a solution of equation (1') cannot have in the interval (a, b) a positive maximum (negative minimum).

Consider the system of equations

$$y'' = A(x)y + (B(x) + c(x)I)y' + F(x), \quad (2)$$

where $y(x)$ and $F(x)$ are n -dimensional column vectors; $A(x)$ and $B(x)$ are square matrices of order n ; $c(x)$ is a scalar function; I is the identity matrix. Put

$$|y|_0 = \sup_x |y(x)|, \quad |A| = \sup_{|\xi|=1} |A\xi|,$$

where ξ is an n -dimensional vector,

$$|A|_0 = \sup_x |A(x)|.$$

Theorem 2. Let $|A(x)|$, $|B(x)|$, and $|c(x)|$ be bounded on every finite interval, and

$$((A - \frac{1}{4}BB^*)\xi, \xi) > 0 \quad (3)$$

for every vector ξ . Then system (2) has no more than one solution bounded for all x .

The proof is based on applying the maximum principle to the equation for $w = \frac{1}{2}(y - z, y - z)$, where y and z are bounded solutions of (2).

Lemma 1. Suppose

$$(A\xi, \xi) \geq a(\xi, \xi), \quad a = \text{const} > 0, \quad (4)$$

$$|F|_0 < +\infty, \quad \frac{1}{4}|B|_0^2 < a, \quad (5)$$

and the quantity $|A(x)|$ is bounded on every finite interval. Then a solution $y = y(x)$ of system (2), bounded for all values of x , satisfies the inequality

$$|y| \leq \frac{|F|_0}{a - \frac{1}{4}|B|_0^2}. \quad (6)$$

The proof is based on applying the maximum principle to the equation satisfied by the function $w = \frac{1}{2}(y, y)$.

Lemma 2. Suppose the quantities $|A|_0$, $|B|_0$, $|F|_0$, $|c|_0$ are finite. Then for the derivative of a bounded solution $y(x)$ of system (2) the estimate

$$|y'|_0 \leq \frac{|A - K^2 I|_0 |y|_0 + |F|_0}{K - |B|_0 + |c|_0}, \quad (7)$$

is valid, where K is any number, $K > |B|_0 + |c|_0$.

Lemma 3. Suppose the system is given

$$y'' = A(x)y + c(x)y' + F(x), \quad (8)$$

where $c(x)$ is a scalar function, $|c|_0 < +\infty$, $A(x)$ is a symmetric matrix, and

$$\alpha(\xi, \xi) \leq (A(x)\xi, \xi) \leq \beta(\xi, \xi), \quad 0 < \alpha \leq \beta. \quad (9)$$

Then system (8) has a unique solution, bounded for all values of x , such that

$$|y| \leq |F|_0 / \alpha. \quad (10)$$

Proof. Uniqueness and inequality (10) follow from Theorem 2 and Lemma 1. Consider a sequence of bounded vector-functions $y_k(x)$ such that $y_0(x) \equiv 0$,

$$y''_{k+1} = \beta y_{k+1} + c(x)y'_{k+1} + F + (A - \beta I)y_k. \quad (11)$$

Such vector-functions exist (3). Applying the maximum principle to the equation for $w_{k+1} = y_{k+1} - y_k$, we obtain

$$|w_{k+1}|_0 \leq \frac{\beta - \alpha}{\beta} |w_k|_0, \quad \text{where} \quad \frac{\beta - \alpha}{\beta} < 1.$$

Consequently, $|w_k|_0 \rightarrow 0$ as $k \rightarrow \infty$. Using Lemma 2 and the equation for w_{k+1} , we prove the convergence of w'_k and w''_k . Hence the assertion of the lemma follows.

Theorem 3. Suppose A is a symmetric matrix satisfying condition (9). Suppose

$$|c|_0 < +\infty, \quad |F|_0 < +\infty, \quad |B|_0 < 2\sqrt{a}. \quad (12)$$

Then there exists a unique solution of system (2), bounded for all values of x , for which inequality (6) and the inequality

$$|y'|_0 \leq \frac{4(\sqrt{\beta} + |c|_0 + |B|_0)^2 - |B|_0^2}{4a - |B|_0^2} \frac{|F|_0}{\sqrt{\beta}} \quad (13)$$

hold.

Proof. Put $K = \sqrt{\beta} + |B|_0 + |c|_0$ in inequality (7). From inequality (9) and the symmetry of A it follows that

$$|A - K^2 I| \leq (\sqrt{\beta} + |B|_0 + |c|_0)^2 - \alpha.$$

Hence, from (6) and (7), inequality (13) follows.

Consider a sequence of bounded vector functions $y_k(x)$ such that

$$y_0 \equiv 0, \quad y_{k+1}'' = A(x)y_{k+1} + c(x)y_{k+1}' + F(x) + B(x)y_k'. \quad (14)$$

By Lemma 3 such a sequence exists. For $w_{k+1} = y_{k+1} - y_k$, by the maximum principle, using Lemma 2, we shall have

$$|w_{k+1}|_0 \leq \frac{|B|_0 |w_k'|_0}{\alpha}, \quad |w_{k+1}'|_0 \leq \frac{|B|_0 (\sqrt{\beta} + |c|_0)^2}{\alpha \sqrt{\beta}} |w_k'|_0. \quad (15)$$

Hence, if $|B|_0 < \alpha \sqrt{\beta} / (\sqrt{\beta} + |c|_0)^2$, we obtain $y_k'(x) \rightarrow y'(x)$, where $y(x)$ is a bounded solution of system (2).

If $|B|_0 \geq \alpha \sqrt{\beta} / (\sqrt{\beta} + |c|_0)^2$, then the theorem can be proved by considering the approximation

$$y_0(x) = 0,$$

$$y_{k+1}'' = (A + LI)y_{k+1} + (B(x) + c(x)I)y_{k+1}' + F(x) - Ly_k,$$

where L is such that

$$|B|_0 < \frac{(\alpha + L)\sqrt{\beta} + L}{(\sqrt{\beta} + L + |c|_0)^2}.$$

Theorem 3 generalizes to the system

$$y'' = f(x, y, y') + c(x)y', \quad (16)$$

where $f(x, y, z)$ is a column vector and $c(x)$ is a scalar function.

Denote by f_y the matrix $\|\partial f_i / \partial y_j\|$, and by f_z the matrix $\|\partial f_i / \partial z_j\|$, $i, j = 1, 2, \dots, n$.

Theorem 4. Suppose that $|c|_0 < +\infty$, f_y is a symmetric matrix,

$$\alpha(\xi, \xi) \leq (f_y \xi, \xi) \leq \beta(\xi, \xi), \quad |f_z| \leq B_0 = \text{const} < 2\sqrt{\alpha}, \quad |f(x, 0, 0)|_0 < +\infty.$$

Then system (16) has a unique solution $y(x)$, bounded for all values of x , such that

$$|y(x)|_0 \leq \frac{|f(x, 0, 0)|_0}{\alpha - \frac{1}{4}B_0^2}, \quad |y'(x)|_0 \leq \frac{4(\sqrt{\beta} + B_0 + |c|_0)^2 - B_0^2}{4\alpha - B_0^2} \frac{|f(x, 0, 0)|_0}{\sqrt{\beta}}.$$

Theorem 5. Suppose that the system

$$y'' = f(x, y, y') \quad (17)$$

is given, where

$$(f_y \xi, \xi) \geq \alpha(\xi, \xi), \quad \alpha = \text{const} > 0, \quad \frac{1}{4}|f_z|^2 \leq \alpha_1 < \alpha,$$

$$\int_0^{+\infty} |f(x, c, 0)|^2 dx < +\infty, \quad c = (c_1, c_2, \dots, c_m) = \text{const}.$$

Then every solution of system (17) bounded on the half-axis $[0, +\infty)$ tends to c as $x \rightarrow +\infty$.

Theorem 6. Suppose that the system

$$y'' = f(x, y) + B(x)y', \quad (18)$$

is given, where

$$(f_y \xi, \xi) \geq \alpha(\xi, \xi), \quad \alpha = \text{const} > 0, \quad B(x) \text{ is a symmetric matrix,}$$

$$\left| \frac{d}{dx} B(x) \right| < 2\alpha, \quad \int_0^{+\infty} |f(x, c)|^2 dx < +\infty, \quad c = \text{const}.$$

Then every solution of system (18) bounded on the half-axis $[0, +\infty)$ tends to c as $x \rightarrow +\infty$.

Theorem 7. Suppose that for system (16) all the conditions of Theorem 4 are satisfied. Suppose, moreover, that the right-hand side of system (16) is a

periodic (almost periodic) function of x for fixed y and z . Then the solution bounded on the whole axis will be periodic (almost periodic).

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