

FACTOR IMAGES OF METRIC SPACES

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Abstract

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MATHEMATICS

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FACTOR IMAGES OF METRIC SPACES

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Definition 1. A topological space X is called **generally metrizable** if on the set $X \times X$ one can define a nonnegative function $\rho(x, y)$ with the following properties: 1) $\rho(x, y) = 0$ if and only if $x = y$; 2) if $U \subset X$, then U is open in X if and only if for every point $x \in U$ there exists $\varepsilon > 0$ such that $U \supset O_\varepsilon^{\rho} x = E\{y : \rho(x, y) < \varepsilon\}$. The function ρ is called a **generalized metric**. If, in addition to 1) and 2), the function ρ also satisfies the condition: 3) $\rho(x, y) = \rho(y, x)$, then it is called a **symmetric**, and the space X is symmetrizable.

Let $f : X \rightarrow Y$ be a given mapping of a metric space X onto a set Y . If ρ is a metric generating the topology of the space X , then on the set Y one can introduce the symmetric $d(y_1, y_2) = \rho(f^{-1}y_1, f^{-1}y_2)$ and consider the topological space $Y_{\rho, f}$ with the topology generated (in the sense of item 2 of Definition 1) by the symmetric d . In ⁽¹⁾, p. 144, A. V. Arkhangel'skii posed the problem of studying all topologies $Y_{\rho, f}$ that can be obtained by varying ρ (of course, without changing the topology of the space X). Theorem 4 of the present article shows that it may be of interest to vary ρ not only in the class of all metrics, but also in the class of all generalized metrics generating the given topology on X . Theorem 5 indicates a sufficiently broad class of mappings for which the symmetrizability of the preimage implies the metrizability of the image; the first three theorems give a characterization of symmetrizable and generally metrizable spaces in the spirit of ⁽³⁾.

Theorem 1. A T_1 -space Y is generally metrizable if and only if there exist: a) a metric space X ; b) its subspace X' ; and c) a continuous mapping $f : X \rightarrow Y$ such that 1) $fX' = Y$; 2) if $U \subset Y$, $U = f(G) = f(G \cap X')$, where G is open in X , then U is open in Y .

Remark. The mapping f is then a quotient mapping.

Proof of Theorem 1. Necessity. Let Y be generally metrizable by the generalized metric ρ . For each natural number n , denote by A_n the discrete space equipotent to Y . Let X be the subspace of the space

$$\prod_{n=1}^{\infty} A_n,$$

consisting of all points $x = \{y_n\}_{n=1}^\infty$, for each of which there is a point $y_x \in Y$ such that: 1) $y_x \in \bigcap_{n=1}^\infty O_{1/n}^\rho y_n$; 2) to each natural number n one can assign an m such that, if $k \geq m$, then $O_{1/n}^\rho y_x \supset O_{1/k}^\rho y_k$. Let X' be the subspace of X consisting of all points all of whose coordinates are equal to one another.

For each $x \in X$, put $fx = y_x$. Since Y is assumed to be a T_1 -space, f is a single-valued mapping. We show that f is continuous. Suppose that in X we have

$$x = \lim_{n \rightarrow \infty} x_n, \quad \text{where } x = \{y^n\}_{n=1}^\infty, \quad x_k = \{y_{ki}\}_{i=1}^\infty$$

($k = 1, 2, \dots$), $fx = y$, $fx_k = y_k$. Let $\varepsilon > 0$. By assumption there exists m such that, if $n \geq m$, then $O_\varepsilon^\rho y \supset O_{1/n}^\rho y^n$. Moreover, since $\lim_{n \rightarrow \infty} x_n = x$, there exists l such that, if $n \geq l$, then $y_{nm} = y^m$. Then, for $n \geq \max(m, l)$,

$$y_n \in O_{1/m}^\rho(y_{nm}) = O_{1/m}^\rho y^m \subset O_\varepsilon^\rho y.$$

Let $U \subset Y$, $U = f(G) = f(G \cap X')$, where G is open in X . Suppose that U is not open in Y , and let $y_0 \in U$ be a point such that for every natural n

$$O_{1/n}^\rho y_0 \cap (Y \setminus U) \neq \Lambda.$$

If $y_n \in O_{1/n}^\rho y_0 \cap (Y \setminus U)$, then the point

$$x_k \{y_0, \underbrace{y_0, \dots, y_0}_{k-1 \text{ times}}, y_k, y_k, \dots, y_k, \dots\}$$

does not belong to the set $f^{-1}U$, and hence, a fortiori, to the set G , since $fx_k = y_k \in Y \setminus U$; on the other hand, if $x_0 = \{y_0, y_0, \dots, \dots, y_0, \dots\}$, then $x_0 \in G$, and, since $p(x_0, x_k) \leq 1/2^*$, for some k one must have $x_k \in G$. The contradiction obtained shows that U is open.

We proceed to prove the **sufficiency** of the condition of Theorem 1. For each point $y \in Y$ choose one point $x_y \in f^{-1}y \cap X'$ and set $Q_{ny} = fO_{1/n}^\rho x_y$. It is clear that, if $U \subset Y$ and U is open, then for any point $y \in U$ there is an $n = n(y, U)$ for which $Q_{ny} \subset U$. Conversely, suppose that for any point $y \in U$ there exists $n = n(y, U)$ for which $Q_{ny} \subset U$. Then

$$f^{-1}U \supset G = \bigcup_{y \in U} O_{1/n(y, U)}^\rho x_y, \quad f(G \cap X') = U.$$

Consequently, by the condition of the theorem, U is open. Thus it is proved that Y satisfies the weak first axiom of countability (see ⁽¹⁾, Definitions 2, 3), and the latter, as is easy to verify, is equivalent to the σ -metrizability of the space Y (see, for example, ⁽⁴⁾, Theorem 1).

Theorem 2. A T_1 -space Y is symmetrizable if and only if there exist: a) a metric space X ; b) its subspace X' , and c) a continuous mapping $f : X \rightarrow Y$

such that: 1) $fX' = Y$; 2) if $U \subset Y$ and $U = f(G) = f(G \cap X')$, where G is open in X , then U is open in Y ; and 3) if U is open in Y and $y \in U$, then for some $\varepsilon > 0$

$$X' \cap O_\varepsilon^\rho f^{-1}y \subset f^{-1}U.$$

Proof. Having Theorem 1, to prove the necessity of Theorem 2 it suffices to make sure that, if the ρ -metric is a symmetric, then for any set U open in Y and for any point $y_0 \in U$ there is an $\varepsilon = \varepsilon(y_0, U) > 0$ such that $O_\varepsilon^\rho f^{-1}y \cap X' \subset f^{-1}U$, where X , X' , and f are taken from the proof of the necessity of Theorem 1. Since U is assumed open, there exists $\varepsilon > 0$ such that $O_\varepsilon^\rho y_0 \subset U$. Then, if $1/(k+1) \leq \varepsilon < 1/k$ and $\eta = 1/2^{k+1}$, it follows from $x \in O_\eta^\rho f^{-1}y \cap X'$ that $x = \{y, y, \dots\}$ and that there exist $\{y_{k+i}\}_{i=2}^\infty$ for which

$$f\{y, y, \dots, y, \underbrace{y}_{k+1 \text{ times}}, y_{k+2}, y_{k+3}, \dots\} = y_0.$$

Consequently, $y_0 \in O_{1/(k+1)}^\rho y$, i.e. $y \in O_\varepsilon^\rho y_0 \subset U$.

To prove the sufficiency, we turn to the proof of sufficiency of the condition of Theorem 1. We show that the sets Q_{ny} constructed for any point $y \in Y$ satisfy the condition: from

$$y \in \bigcap_{n=1}^\infty Q_n y_n$$

it follows that $y = \lim y_n$. Indeed, if U is open and $y \in U$, then there exists k such that for any $n \geq k$

$$O_{1/n}^\rho f^{-1}y \cap X' \subset f^{-1}U.$$

Then, since the condition $y \in Q_n y_n$ is equivalent to the condition

$$f^{-1}y \cap O_{1/n}^\rho x_{y_n} \neq \Lambda,$$

we obtain

$$x_{y_n} \in O_{1/n}^\rho f^{-1}y \cap X' \subset f^{-1}U,$$

i.e. $y_n \in U$. After this, the proof of Theorem 2 can be completed by reference to Theorem 3 of ⁽⁴⁾.

Definition 2 (see ⁽¹⁾, p. 140, or 3, def. 1). A mapping $f : X \rightarrow Y$ of a space X , ρ -metrized by an ρ -metric, onto a topological space Y is called a P -mapping if, for every open in

* For any two points $x = \{x_n\}_{n=1}^\infty \in X$ and $z = \{z_n\}_{n=1}^\infty \in X$, $p(x, z) = 1/2^k$, where

$$k = \min\{n : x_n \neq z_n\}.$$

of the set U and for any point $y \in U$ there is an $\varepsilon > 0$ such that $O_\varepsilon^\rho f^{-1}y \subset f^{-1}U$.

Theorem 3. A T_1 -space Y is symmetrizable by a symmetric satisfying condition (AH)* if and only if there exist: a) a metric space X ; b) its subspace X' and c) a continuous mapping $f : X \rightarrow Y$ such that: 1) $fX' = Y$; 2) if $U \subset Y$, $U = f(G) = f(G \cap X')$, where G is open in X , then U is open in Y , and 3) f is a Π -mapping (with respect to some metric generating the topology of the space X).

Proof. Necessity. Let the space Y be symmetrizable by a symmetric ρ satisfying condition (AH). To any point $y \in Y$ and to each natural number n we assign the set $Q_{ny} = O_{1/n}^\rho y$, such that $\text{diam}(O_{1/(n-1)}^\rho y) \geq 1/n$, and $\text{diam} Q_{ny} < 1/n$. For each natural n , let A_n denote the discrete space of the same cardinality as Y , and let X be the subspace of the space

$$\prod A_n,$$

consisting of all points $x = \{y_1, y_2, \dots, y_n, \dots\}$ for which there exist points $y_x \in Y$ such that $y_x \in \bigcap_1^\infty Q_{ny_n}$. Finally, let X' be the subspace of X consisting of all points with constant coordinates. For each point $x \in X$ put $fx = y_x$. By arguments analogous to those used in the proof of Theorem 1, it is easy to show that the mapping f satisfies the first two conditions of Theorem 3.

Let us show that f also satisfies the third condition. Let U be open in Y and $y_0 \in U$. Then there exists a natural number n such that $O_{1/n}^\rho y_0 \subset U$. If $x \in O_{1/2}^\rho f^{-1}y_0$, then there exist $\{y^i\}_{i=n+1}^\infty$ such that, if $x = \{y_k\}_{k=1}^\infty$, then

$$f(y_1, y_2, \dots, y_n, y^{n+1}, y^{n+2}, \dots) = y_0.$$

Consequently, $y_0 \in Q_n y_n$, $fx \in Q_n y_n$, i.e. $\rho(y_0, fx) < 1/n$, whence $fx \in O_{1/n}^\rho y_0 \subset U$. Thus the necessity of the condition of Theorem 3 is proved.

Sufficiency. For any point $y \in Y$ choose a point $x_y \in f^{-1}y \cap X'$ and put $Q_{ny} = fO_{1/n}^\rho x_y$. Put also $\omega_n = \{Q_{ny}\}_{y \in Y}$ and $Q'_n y = St_{\omega_n} y$. On the set Y consider the following two ρ -metrics:

$$\rho(y_1, y_2) = \begin{cases} 2, & \text{if } y_2 \notin Q_1 y_1, \\ 1/2^n, & \text{if } y_2 \in Q_n y_1 / Q_{n+1} y_1, \\ 0, & \text{if } y_1 = y_2; \end{cases}$$

$$\rho'(y_1, y_2) = \begin{cases} 2, & \text{if } y_2 \notin Q'_1 y_1, \\ 1/2^n, & \text{if } y_2 \in Q'_n y_1 / Q'_{n+1} y_1, \\ 0, & \text{if } y_1 = y_2. \end{cases}$$

It is clear that the space Y is ρ -metrizable by the ρ -metric ρ and that the ρ -metric ρ' is a symmetric. We show that Y is symmetrizable by the symmetric ρ' . Indeed, let U be open in Y and let $y_0 \in U$. For some n , $O_{1/n}^\rho f^{-1}y_0 \subset f^{-1}U$. Let $y \in O_{1/4^n}^{\rho'} y_0$. Consequently, $y \in Q_{2n} z$, where z is such that $y_0 \in Q_{2n} z$. The latter inclusion gives $O_{1/2n}^\rho x_z \cap f^{-1}y_0 \neq \emptyset$, whence we conclude

that $O_{1/2n}^\rho x_z \subset O_{1/n}^\rho f^{-1}y_0$. Consequently, $f^{-1}y \cap O_{1/n}^\rho f^{-1}y_0 \neq \Lambda$, which gives $y \in U$. If, conversely, U contains each of its points together with some spherical neighborhood with respect to ρ' , then U is open, since then U contains each of its points together with some spherical neighborhood with respect to ρ . Thus, the space Y is symmetrizable by the symmetric ρ' . Let now a sequence of points

* See ⁽¹⁾, p. 147. A symmetric ρ satisfies condition (AH) if, whatever the point $x \in X$ and $\varepsilon > 0$, there exist

$\{y_n\}_{n=1}^\infty$ converges to the point y , and let $\varepsilon > 0$. For sufficiently large n we have $\rho(y, y_n) < 1/2^k \leq \varepsilon$, i.e., $y_n \in Q_{k+1}y$, and, consequently, for such (sufficiently large) m and n , $\rho'(y_m, y_n) < \varepsilon$. All this means that the semimetric ρ' satisfies condition (AH).

Theorem 4. A T_1 -space Y is o -metrizable if and only if there exist: a) a metric space X ; b) an o -metric $\tilde{p}$ on X consistent with the topology of X ; and c) a quotient mapping $f: X \rightarrow Y$ which is a Π -mapping with respect to $\tilde{p}$. Moreover, if Y is o -metrizable, then for any of its o -metrics ρ there exist corresponding $X = X_\rho$, $f = f_\rho$, $\tilde{p} = \tilde{p}_\rho$ such that

$$\rho(y_1, y_2) = \tilde{p}(f^{-1}y_1, f^{-1}y_2)$$

for any two points $y_1, y_2 \in Y$.

Proof. Necessity. Let Y be o -metrizable with o -metric ρ , and let X , its metric p , and the mapping f be the same as in the proof of Theorem 1. Consider on X the o -metric $\tilde{p}$ defined as follows:

$$\tilde{p}(x_1, x_2) = p(x_1, x_2) - p(f^{-1}fx_1, f^{-1}fx_2) + \rho(fx_1, fx_2).$$

We first show that $\tilde{p}$ generates on X the same topology as p . Let

$$\lim_{n \rightarrow \infty} p(x_0, x_n) = 0.$$

Then

$$\lim_{n \rightarrow \infty} p(f^{-1}fx_0, f^{-1}fx_n) = 0,$$

since

$$p(f^{-1}fx_0, f^{-1}fx_n) \leq p(x_0, x_n),$$

and

$$\lim_{n \rightarrow \infty} \rho(fx_0, fx_n) = 0,$$

since f is continuous. Hence

$$\lim_{n \rightarrow \infty} \tilde{p}(x_0, x_n) = 0.$$

Conversely, let

$$\lim_{n \rightarrow \infty} \tilde{p}(x_0, x_n) = 0.$$

Then

$$\lim_{n \rightarrow \infty} \rho(fx_0, fx_n) = 0,$$

and hence, by the definition of the mapping f ,

$$\lim_{n \rightarrow \infty} p(f^{-1}fx_0, f^{-1}fx_n) = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} p(x_0, x_n) = 0.$$

The fact that $\rho(y_1, y_2) = \tilde{p}(f^{-1}y_1, f^{-1}y_2)$ and that f is a Π -mapping with respect to $\tilde{p}$ follows directly from the definition of $\tilde{p}$.

Sufficiency. Let X be ρ -metrizable with ρ -metric $\tilde{p}$, and let the mapping $f : X \rightarrow Y$ be quotient and, with respect to $\tilde{p}$, a Π -mapping. On the set Y introduce the ρ -metric

$$\rho(y_1, y_2) = \tilde{p}(f^{-1}y_1, f^{-1}y_2)$$

and show that it generates the topology of the space Y . Indeed, if U is open in Y and $y_0 \in U$, then from

$$\varepsilon = \tilde{p}(f^{-1}y_0, X \setminus f^{-1}U) > 0$$

it follows that $O_\varepsilon^\rho y_0 \subset U$. Conversely, suppose that the set U contains all its points together with their spherical neighborhoods with respect to ρ , and let $x_0 \in f^{-1}U$. If

$$O_\varepsilon^\rho fx_0 \subset U,$$

then

$$O_\varepsilon^{\tilde{p}} x_0 \subset f^{-1}U,$$

and, consequently, $f^{-1}U$ is open in X . Since by assumption f is quotient, U is open in Y . The theorem is proved.

We note that if, in the proof of Theorem 4, the words “ ρ -metrizable” and “ ρ -metric $\tilde{p}$ ” are replaced by: a) “symmetrizable” and “symmetric $\tilde{p}$,” or by b) “symmetrizable with a symmetric satisfying the weak Cauchy condition” and “a symmetric $\tilde{p}$ satisfying the weak Cauchy condition,” then true assertions are obtained, which can be proved in the same way.

Definition 3. A mapping $f : X \rightarrow Y$ from an ρ -metrizable space X with ρ -metric ρ onto a topological space Y is called K -uniform (with respect to ρ) if for every bicomact set $\Phi \subset Y$ and every neighborhood U of it the condition

$$\rho(f^{-1}\Phi, X \setminus f^{-1}(U)) > 0$$

is satisfied.

Theorem 5. A T_2 -space is metrizable if and only if it is a quotient K -uniform (with respect to some symmetric) image of a symmetrizable space.

The proof of Theorem 5 repeats exactly the arguments of A. V. Arhangel'skii given in the proof of Theorem 3.3 of ⁽¹⁾.

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³ R. W. Heath, *Fund. Math.*, **57**, 1, 91 (1965).

⁴ S. I. Nedeв, *Dokl. Bulgarsk. Akad. Nauk*, **20**, no. 6, 513 (1967).

* See ⁽²⁾. A symmetric of a space X satisfies the weak Cauchy condition if, from the fact that A is not closed in X , it follows that for every $\varepsilon > 0$ there are two points x and y in A such that $\rho(x, y) < \varepsilon$ and $x \neq y$.

Note: Figure translations are in progress. See original paper for figures.

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