

ON MAPPINGS OF ZERO-DIMENSIONAL BICOMPACTS

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1968

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Abstract

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UDC 513.831 + 519.5

MATHEMATICS

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ON MAPPINGS OF ZERO-DIMENSIONAL BICOMPACTS

(Presented by Academician P. S. Aleksandrov on 30 III 1967)

In this note a sufficient condition is given under which a zero-dimensional bi-compact R is mapped onto the generalized Cantor discontinuum D^τ .

A system $\mathfrak{B} = \{\bar{U}_\alpha\}$, consisting of canonically closed sets, will be called a δ -system of a closed set F of a topological space R , if the following two conditions are satisfied: 1) for any finite collection $\bar{U}_{\alpha_1}, \dots, \bar{U}_{\alpha_s}$ belonging to the system $\mathfrak{B}$, we have

$$\text{int } \bar{U}_{\alpha_1} \cap \dots \cap \text{int } \bar{U}_{\alpha_s} \neq \emptyset;$$

2) $F = \bigcap_\alpha \bar{U}_\alpha$ for all $\bar{U}_\alpha \in \mathfrak{B}$.

We shall call a system $\mathfrak{B}$ *small* if for any open $V \supset F$ there exists $\bar{U}_{\alpha_0} \in \mathfrak{B}$ such that $F_0 \subset \bar{U}_{\alpha_0} \subset V$. The minima of the cardinalities of (small) δ -systems will be called the δ -pseudocharacter (δ -character) of the set F and will be denoted respectively by $\delta_\psi(F, R)$ or $\delta(F, R)$. We note that if R is bicomact, then $\delta_\psi(F, R) = \delta(F, R)$ for any closed set $F \subset R$. Further, if the character of a closed set $F \subset R$ is denoted by $\chi(F, R)$, then always $\delta(F, R) \leq \chi(F, R)$. However, since F need not belong to $\text{int } \bar{U}_\alpha$ for any $\bar{U}_\alpha \in \mathfrak{B}$, even for bicomacts a strict inequality $\delta(F, R) < \chi(F, R)$ is possible. Nevertheless the following proposition is valid:

If $R = \prod X_\alpha$ is the topological product of compact metrizable spaces X_α , then for any closed $F \subset R$ we have $\delta(F, R) = \chi(F, R)$.

A limiting cardinal number $\mathfrak{m} = \lim_\alpha \mathfrak{n}_\alpha$, $\mathfrak{n}_\alpha < \mathfrak{m}$, will be called *singular*. If $\mathfrak{m} = \aleph_\zeta$, then ζ is a limiting ordinal number. The cardinal number $\mathfrak{m}^+$, immediately following the singular number $\mathfrak{m}$, will be called *metasingular*. Thus, $\mathfrak{m}^+ = \aleph_{\zeta+1}$. Every cardinal number $\mathfrak{n}$ of the form $\aleph_{\zeta+2}$ will be called *simple*. By $\log \mathfrak{m}$ we shall denote the least cardinal number $\mathfrak{n}$ such that $2^\mathfrak{n} = \exp \mathfrak{n} \geq \mathfrak{m}$. The cardinality of a set A is denoted by $|A|$. If ξ is an ordinal number, then $|\xi|$ denotes the cardinality of the set of all ordinal numbers $< \xi$. By wR , sR , and cR are denoted respectively the weight, the density, and the cellularity of R . Theorems proved with the aid of the generalized continuum hypothesis are marked by the letters g.c.h.

Theorem 1 (g.c.h.). *If for every point x of a zero-dimensional bicomact R the inequality $\delta(x, R) \geq \tau \geq \aleph_0$ holds, then R is continuously mapped onto D^τ , if τ is not a metasingular number, or onto $D^{\log \tau}$, if τ is a metasingular number.*

The proof of this theorem is based on the following two lemmas.

Lemma 1. *Let R be a bicomact and let for every point $x \in R$ we have $\delta(x, R) \geq \tau \geq \aleph_0$. Let F be a closed subset of R and $\delta(F, R) \leq \mathfrak{m} < \tau$. Then $|F| \geq \exp \tau$.*

Proof. We shall first show that for every point $x \in F$ we have $\chi(x, F) \geq \tau$. Indeed, if for some point $x_0 \in F$ we had $\chi(x_0, F) \leq \mathfrak{n} < \tau$, then

$$x_0 = \bigcap_{\alpha \in A} O_\alpha \cap F,$$

where O_α are open sets in R . For each $\alpha \in A$ consider such a $\bar{U}_\alpha$ that $x_0 \in \text{int } \bar{U}_\alpha \subset$

$\subset O_\alpha$. In this case

$$x_0 = \bigcap_{\alpha \in A} \bar{U}_\alpha.$$

Since $\delta(F, R) \leq \mathfrak{m}$, there exists a δ -system $\mathfrak{B} = \{\bar{V}_\alpha\}$ such that

$$F = \bigcap_{\alpha \in B} \bar{V}_\alpha, \quad |B| \leq \mathfrak{m}.$$

Thus we obtain

$$x_0 = \bigcap_{\alpha \in A} \bar{U}_\alpha \cap \bigcap_{\alpha \in B} \bar{V}_\alpha$$

and $|A| + |B| = \mathfrak{n} + \mathfrak{m} < \tau$, while the family $\{\bar{U}_\alpha\} \cup \{\bar{V}_\alpha\}$ is a δ -system, since $x_0 \in \text{int } \bar{U}_\alpha$ for all $\alpha \in A$. Hence $\delta(x_0, R) < \tau$, contrary to the assumption. Thus, for every point $x \in F$ we have $\chi(x, F) \geq \tau$. From the Čech-Pospíšil theorem ⁽¹⁾ it follows that $|F| \geq \exp \tau$. The lemma is proved.

Lemma 2. Let R be a zero-dimensional bicomactum, and suppose that for each point of it we have $\delta(x, R) \geq \tau \geq \aleph_0$. Let Φ be a closed subset of R and $|\Phi| \leq \tau$. Further, let $f: R \rightarrow D^{\mathfrak{m}}$ be a continuous mapping onto the generalized Cantor discontinuum of weight $\mathfrak{m}$, with $\mathfrak{m}^{\aleph_0} < \tau$. Then there exists a clopen neighborhood $U' = U(\Phi)$ of the set Φ such that $U'' = R - U'$ has the property

$$U'' \cap f^{-1}y \neq \emptyset$$

for all $y \in D^{\mathfrak{m}}$.

Proof. Denote $F_y = f^{-1}y$. Suppose the contrary. This means that for every clopen neighborhood $U_\alpha(\Phi) = U_\alpha$ there is an $F_y \subset U_\alpha$. Denote by V_α the set of all F_y lying in U_α . Since the mapping f is continuous and closed, V_α is open. Further, for every α we have $\bar{V}_\alpha \cap \Phi = \emptyset$, for otherwise we would find an $F_y \subset U_\alpha \setminus \bar{V}_\alpha$, contradicting the definition of V_α . We now note that the number of sets

of the form $f^{-1}G_\alpha = \bar{V}_\alpha$ is equal to the number of all canonically closed sets $\bar{G}_\alpha$ in D^m , i.e. $\mathfrak{m}^{N_0}$ (see (2)). Consider the set of all such $\bar{V}_\alpha$, and let $\{\bar{V}_\alpha\} = \mathfrak{B}$. By what has just been said, $|\mathfrak{B}| \leq \mathfrak{m}^{N_0} < \tau$. We show that the system $\mathfrak{B}$ is a δ -system. Consider an arbitrary finite number of elements $\bar{V}_{\alpha_1}, \dots, \bar{V}_{\alpha_s}$ belonging to $\mathfrak{B}$; for each of them consider the corresponding $U_{\alpha_1}, \dots, U_{\alpha_s}$. Let

$$U_{\alpha_1 \dots \alpha_s} = U_{\alpha_1} \cap \dots \cap U_{\alpha_s}$$

be a clopen set. Consequently, by the assumption, there exists $F_y \subset U_{\alpha_1 \dots \alpha_s}$, and hence also an entire neighborhood of the form $f^{-1}G$, where G is an open set in D^m , but $f^{-1}G$ belongs to $V_{\alpha_1}, \dots, V_{\alpha_s}$, i.e.

$$\text{int}(\bar{V}_{\alpha_1} \cap \dots \cap \bar{V}_{\alpha_s}) \neq \emptyset.$$

Thus the system $\mathfrak{B}$ is a δ -system; consequently, for

$$L = \bigcap_{\alpha} \bar{V}_\alpha$$

we have

$$\delta_\Phi(L, R) = \delta(L, R) = |\mathfrak{B}| < \tau.$$

Applying Lemma 1, we obtain $|L| \geq \exp \tau$. This contradicts the fact that $L \subset \Phi$, and consequently $|L| \leq |\Phi| \leq \tau$. The lemma is proved.

Proof of Theorem 1. We shall carry out the proof of this theorem by transfinite induction. Namely, we construct a system of binary coverings $\omega_\alpha = \{A_\alpha^0, A_\alpha^1\}$, $\alpha \in A$, of the bicomactum R , such that $|A| = \tau$ in the first case, or $|A| = \log \tau$ in the second case, and: 1) $A_\alpha^0 \cup A_\alpha^1 = R$; 2) $A_\alpha^0 \cap A_\alpha^1 = \emptyset$; 3) for any set of indices $\alpha_1, \dots, \alpha_s$ and the corresponding set of zeros and ones $i_1, \dots, i_s$ we have

$$A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \neq \emptyset.$$

In that case the system of characteristic functions $\{\varphi_\alpha\}$: $\varphi_\alpha(A_\alpha^0) = 0$ and $\varphi_\alpha(A_\alpha^1) = 1$, defines a natural mapping $f: R \rightarrow D^{|A|}$ by the rule

$$f(x) = \{\varphi_\alpha(x)\} \in D^{|A|}$$

(the fact that the mapping is “onto” follows from the centeredness of $\{A_\alpha^{i(\alpha)}\}$ for distinct α). We shall construct such a system of coverings $\{\omega_\alpha\}$ by transfinite induction. For (A_1^0, A_1^1) we choose an arbitrary clopen disjoint pair covering R . Suppose that the pairs

$$(A_1^0, A_1^1), \dots, (A_\alpha^0, A_\alpha^1), \dots, \quad \alpha < \xi < \omega(\tau),$$

have been constructed, for all ordinal numbers α less than some $\xi < \omega(\tau)$, so that for all $\alpha < \xi$ we have: 1) $A_\alpha^0 \cup A_\alpha^1 = R$; 2) $A_\alpha^0 \cap A_\alpha^1 = \emptyset$; 3) for all $\alpha_1, \dots, \alpha_s < \xi$ and any set $i_1, \dots, i_s$ of zeros or ones we have

$$A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \neq \emptyset.$$

Using the distributivity of intersection with respect to union of sets, we obtain

$$R = \bigcap_{\alpha < \xi} (A_\alpha^0 \cup A_\alpha^1) = \bigcup_{i(\alpha)} \bigcap_{\alpha < \xi} A_\alpha^{i(\alpha)} = \bigcup_x F_x, \quad x \in D^{|\xi|}, \quad (1)$$

where $F_x = \bigcap_{\alpha < \xi} A_\alpha^{i(\alpha)}$, and x is an arbitrary sequence of zeros and ones on the set of all ordinal numbers $< \xi$. Moreover, the decomposition R by formula (1) is a continuous partition generated by the continuous mapping $\varphi_\xi : R \rightarrow D^{|\xi|}$ according to the rule $\varphi_\xi(F_x) = \{i_\alpha\} = x \in D^{|\xi|}$. Consequently, $F_x = \varphi^{-1}(x)$. Consider the cardinal number $|\xi|$. Since $\xi < \omega(\tau)$, we have $|\xi| < \tau$. We shall prove that, in the case where τ is singular or regular, the induction step is possible for every $\xi < \omega(\tau)$, while if τ is metasingular, the induction step is possible only for $\xi < \omega(\log \tau)$.

Case 1 (τ singular). Consider in $D^{|\xi|}$ a dense subset S of cardinality $|S| = |\xi| = \aleph_\beta < \tau$. Choose, for each point $x_s \in \varphi_\xi^{-1}(s)$, $s \in S$, and put $M = \bigcup_{s \in S} \overline{x_s}$. Then $|M| \leq \exp \exp \aleph_\beta = \aleph_{\beta+2} < \tau$ (τ is singular!) and $M \cap F_x \neq \emptyset$ for every $x \in D^{|\xi|}$, which follows from the closedness of the mapping φ_ξ and the bicomactness of R .

Case 2 ($\tau = \aleph_{\beta+2}$ regular). Since $|\xi| < \tau$, we have $|\xi| \leq \aleph_{\beta+1}$. Since $\exp \aleph_\beta = \aleph_{\beta+1}$, the space $D^{|\xi|}$ contains a dense subset S of cardinality $\aleph_\beta$ (see (3)). Constructing, as in the first case, the set M , we obtain that $|M| \leq \exp \exp \aleph_\beta = \aleph_{\beta+2} = \tau$.

Case 3 ($\tau = \aleph_{\beta+1}$ and $\beta = \lim \alpha$ is metasingular). In this case $|\xi| < \log \tau = \aleph_\beta$, where $\aleph_\beta$ is a singular cardinal. Thus, as in the first case, we choose the set S and construct the closed set $M \subset R$.

Thus, in all three cases there exists in R a closed set M of cardinality $\leq \tau$ (or $\log \tau$), and moreover $M \cap F_x \neq \emptyset$ for all $x \in D^{|\xi|}$. We use Lemma 2. For this, note that $|\xi|^{\aleph_0} < \tau$, since $|\xi| < \tau$ (or $\log \tau$), and τ is a singular or regular (or metasingular) cardinal. From Lemma 2 it follows that there exists a clopen neighborhood $U^0(M) = U^0$ such that $R \setminus U^0 = U^1$ has the property that $U^1 \cap F_x \neq \emptyset$ for all $x \in D^{|\xi|}$. At the same time note that $F_x \cap U^0 \neq \emptyset$ for all $x \in D^{|\xi|}$, since $U^0 \supset M$. Next, it is obvious that $U^0 \cap U^1 = \emptyset$ and $U^0 \cup U^1 = R$. We take the pair (U^0, U^1) as the required pair (A_ξ^0, A_ξ^1) . The induction step is completed.

We shall prove that property 3) is fulfilled. Let $\alpha_1, \dots, \alpha_s, \xi$ be an arbitrary set of indices $\alpha_i \leq \xi$, and $i_1, \dots, i_s, i_{s+1}$ an arbitrary set of zeros and ones. Note that

$$A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \supset F_{x_0},$$

where x_0 is the point in $D^{|\xi|}$ having, in the places $\alpha_1, \dots, \alpha_s$, the values $i_1, \dots, i_s$. But, by construction,

$$F_{x_0} \cap (A_\xi^{i_{s+1}} = U^i) \neq \emptyset,$$

if i is equal to zero or one. Consequently,

$$A_{\alpha_1}^{i_1} \cap \dots \cap A_{\alpha_s}^{i_s} \cap A_{\xi}^{i_{s+1}} \neq \emptyset.$$

Property 3) is proved. Thus, using the principle of transfinite induction, for every $\xi < \tau$ (or $\log \tau$) we construct the system of binary open coverings mentioned above. Hence, as was already noted, there follows the existence of a continuous mapping $f : R \rightarrow D^\tau$ (or onto $D^{\log \tau}$). Theorem 1 is completely proved.

Theorem 2. *If a bicomactum R is mapped continuously onto the generalized Cantor discontinuum D^τ , $\tau \geq \aleph_0$, then R contains no fewer than $\exp \tau$ points x , at each of which $\chi(x, R) \geq \tau$.*

The proof of this theorem follows from the following Lemma 3, which is a special case of a more general theorem proved by I. Juhász ⁽⁴⁾.

Lemma 3. *Let $f : R \rightarrow D^\tau$ be a continuous mapping of a bicomactum R onto D^τ . Then for every $x \in D^\tau$ there exists a point $y \in f^{-1}x$ such that $\chi(y, R) \geq \tau$.*

I do not know whether in Theorem 2 one can replace $\chi(x, R)$ by $\delta(x, R)$. In any case, in Lemma 3 this cannot be done. Let us further note that in the formulation of Theorem 1 one cannot replace the inequality $\delta(x, R) \geq \tau$ by the inequality $\chi(x, R) \geq \tau$. This follows from the fact that for every $\tau \geq \aleph_0$ there exist

there exists a zero-dimensional ordered bicomactum X , and moreover $\chi(x, X) \geq \tau$ for all $x \in X$. If it were possible to map X onto D^τ , then, by the theorem of S. Mardešić and P. Papić ⁽⁵⁾, it would follow that D^τ is metrizable, i.e. $\tau \leq \aleph_0$.

Theorem 3 (GCH). Let R be a zero-dimensional bicomactum. If for every point $x \in R$ the inequality $\delta(x, R) \geq \tau$ holds, then R topologically contains all extremally disconnected spaces of weight $\leq \tau$ (τ nonmeasurable) or of weight $\leq \log \tau$ (τ measurable).

The proof of this theorem is based on Theorem 1 and on a theorem of V. I. Ponomarev ⁽⁶⁾ stating that every irreducible perfect preimage of an extremally disconnected space X is homeomorphic to X .

Theorem 4 (GCH). Let R be an extremally disconnected bicomactum. Then:

- 1) If $wR = \aleph_{\alpha+3}$, then R maps continuously onto $D^{\aleph_\alpha}$.
- 2) If $wR = \tau$ or τ^+ or τ^{++} , where τ is a singular cardinal, then R maps continuously onto $D^{\mathfrak{m}}$ for every $\mathfrak{m} < \tau$.
- 3) If R is a bicomactum homogeneous with respect to character of weight τ , then R maps continuously onto $D^{\log \tau}$ for every τ .

The proof of this theorem is based on Theorem 1 and on the following two lemmas.

Lemma 4 (A. Hajnal, I. Juhász ⁽⁷⁾). Let R be a Hausdorff space and $\mathfrak{m}$ an infinite cardinal number. If the set of those points $x \in R$ for which $\chi(x, R) \leq \mathfrak{m}$ has cardinality greater than $\exp \mathfrak{m}$, then R contains more than $\mathfrak{m}$ pairwise disjoint open subsets.

Lemma 5. Let R be an extremally disconnected bicompactum containing $\mathfrak{m} \geq \aleph_0$ pairwise disjoint open sets. Then R maps continuously onto $D^{\exp \aleph}$.

Corollary 1. Every extremally disconnected bicompactum homogeneous with respect to character, of weight τ , if τ is singular, has cardinality $\exp \tau$.

Corollary 2. Every complete Boolean algebra of large cardinality contains a free subalgebra of sufficiently large cardinality. The estimates of this cardinality are the same as in Theorem 4.

The author does not know whether the estimates of the weight of D^n in Theorem 4 can be improved.

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Received
22 III 1967

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Note: Figure translations are in progress. See original paper for figures.

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