

FIRST-ORDER EQUATIONS FOR THE GRAVITATIONAL FIELD

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Abstract

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FIRST-ORDER EQUATIONS FOR THE GRAVITATIONAL FIELD

The basis of the general theory of relativistic wave equations describing elementary particles is the standard form of the equation

$$(\gamma^k \partial_k + \gamma^0) \psi = 0, \quad k = 1, 2, 3, 4. \quad (1)$$

Here ψ is a multicomponent function; γ^k, γ^0 are constant square matrices. For $|\gamma^0| = 0$, equation (1) describes particles with zero rest mass ⁽¹⁾. Important properties of system (1) are the following:

1. Any system of linear homogeneous differential equations of arbitrary order with constant coefficients can be reduced to this form, provided that the number of unknown functions is equal to the number of equations; i.e., for such equations the form (1) is completely general and universal.
2. System (1) is completely and uniquely determined by specifying the matrices γ^k, γ^0 . Therefore, knowing the algebraic properties of the matrices γ^k, γ^0 , one can obtain much information about the properties of the solutions of system (1), which is used in the broadest way in the theory of elementary particles (see, for example, ⁽²⁾).

For **nonlinear** differential equations with constant coefficients, in which the unknown functions and their derivatives enter rationally, there also exists a universal form of the type

$$(\gamma^k \partial_k + \gamma^0)_{\mu\nu} \psi_\nu + \alpha_{\mu\nu\lambda} \psi_\nu \psi_\lambda = 0, \quad \mu, \nu, \lambda = 1, 2, \dots, n. \quad (2)$$

Indeed, derivatives of any order $\partial^s \psi_\mu / \partial x_k^s$, including mixed ones, by successive substitutions

$$\psi'_\mu = \partial \psi_\mu / \partial x_k, \quad \psi''_\mu = \partial \psi'_\mu / \partial x_k, \dots \quad (3)$$

can be reduced to first-order derivatives, while products $\psi_{\mu_1} \psi_{\mu_2} \psi_{\mu_3} \dots$, by the substitutions

$$\chi' = \psi_{\mu_1} \psi_{\mu_2}, \quad \chi'' = \chi' \psi_{\mu_3}, \dots \quad (4)$$

are reduced to products of no more than two functions. Since relations of the form (3), (4) are special cases of equations (2), the resulting system will have the same form. In particular, the known nonlinear spinor equations of quantum field theory, as well as the system of equations for interacting electromagnetic and electron-positron fields in quantum electrodynamics, belong to type (2).

Equations (2), analogously to (1), along with broad universality, are characterized by the fact that their properties and the properties of the fields they describe are determined to a significant extent by the algebraic properties of the constant matrices γ^k, γ^0 and the third-rank matrix $\alpha_{\mu\nu\lambda}$.

Einstein's equations of the gravitational field can also be reduced to the universal form (2). For empty space these equations have the form ^(3,4)

$$R_{ik} = \partial_k \Gamma_{il}^l - \partial_l \Gamma_{ik}^l + \Gamma_{il}^m \Gamma_{kn}^l - \Gamma_{ik}^l \Gamma_{ln}^m = 0. \quad (5)$$

Here $R_{ik} = R_{ki}$ is the Ricci tensor, $\Gamma_{ik}^l = \Gamma_{ki}^l$ are the Christoffel symbols of the second kind,

$$\Gamma_{ik}^l = 1/2 g^{ln} (\partial_i g_{kn} + \partial_k g_{in} - \partial_n g_{ik}). \quad (6)$$

Equation (5) is a special case of the equation for Einstein spaces ^(4,5)

$$R_{ik} = \lambda g_{ik}. \quad (7)$$

Let us write equations (6), (7) in the form

$$\partial_l \Gamma_{ik}^l - \partial_k \Gamma_{il}^l + \lambda g_{ik} + \Gamma_{ik}^l \Gamma_{ln}^n - \Gamma_{il}^n \Gamma_{kn}^l = 0, \quad (8)$$

$$\partial_l g_{ik} - \partial_i g_{kl} - \partial_k g_{il} + 2g_{ln} \Gamma_{ik}^n = 0. \quad (9)$$

If the set of the 10 quantities g_{ik} and the 40 quantities Γ_{ik}^l is regarded as the components of a single function*

$$\psi = \begin{pmatrix} \psi_{(ik)} \\ \psi_{l(ik)} \end{pmatrix} = \begin{pmatrix} g_{ik} \\ \Gamma_{ik}^l \end{pmatrix} = \begin{pmatrix} g \\ \Gamma \end{pmatrix}, \quad (10)$$

then the 50 equations (8), (9) can be represented as a single system of type (2)

$$(\gamma^j \partial_j + \lambda \gamma^0) \psi + \psi \Lambda \psi = 0. \quad (11)$$

The block structure of the matrices γ^j and γ^0 is clear from (8)–(10):

$$\gamma^j = \begin{pmatrix} 0 & \gamma_{g\Gamma}^j \\ \gamma_{\Gamma g}^j & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1_g & 0 \\ 0 & 0 \end{pmatrix}. \quad (12)$$

Here 1_g is the unit matrix in the 10-dimensional subspace of the independent components of the metric tensor g_{ik} . It is interesting to note that γ^j and γ^0 have the same general structure as the matrices of the generalized relativistic wave equations (1), and satisfy the same relations (see ⁽⁶⁾):

$$\gamma^0 \gamma^j + \gamma^j \gamma^0 = (1 - \gamma^0) \gamma^j + \gamma^j (1 - \gamma^0) = \gamma^j,$$

$$\gamma^0 (\gamma^0 - 1) = 0, \quad \gamma^0 \gamma^j \gamma^0 = (1 - \gamma^0) \gamma^j (1 - \gamma^0) = 0. \quad (13)$$

The symbol Λ denotes a set of 50 matrices of order 50 (a matrix vector)

$$\Lambda = \begin{pmatrix} \Lambda^{(ik)} \\ \Lambda^{l(ik)} \end{pmatrix} = \begin{pmatrix} \Lambda^g \\ \Lambda^\Gamma \end{pmatrix}. \quad (14)$$

From (8)–(10) the block structure of the matrices Λ^g and Λ^Γ is also clear:

$$\Lambda^g = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{\Gamma\Gamma}^g \end{pmatrix}, \quad \Lambda^\Gamma = \begin{pmatrix} 0 & \Lambda_{g\Gamma}^\Gamma \\ \Lambda_{\Gamma g}^\Gamma & 0 \end{pmatrix}, \quad \tilde{\Lambda}_{g\Gamma}^\Gamma = \Lambda_{\Gamma g}^\Gamma. \quad (15)$$

Equations (11) can now be written in the form of two systems, corresponding to (8) and (9):

$$\gamma_{g\Gamma}^j \partial_j \Gamma + \lambda g + \Gamma \Lambda_{\Gamma\Gamma}^g \Gamma = 0, \quad (8')$$

$$\gamma_{\Gamma g}^j \partial_j g + g \Lambda_{g\Gamma}^\Gamma \Gamma + \Gamma \Lambda_{\Gamma g}^\Gamma g = 0. \quad (9')$$

It is not difficult to write out explicitly the matrices γ^j , γ^0 , $\Lambda^{(ik)}$, $\Lambda^{l(ik)}$ by means of matrix algebra in the space of the functions ψ . As is known, this algebra is usually formed by matrices e^{AB} , all of whose elements are zero except for one, equal to unity, situated at the inter—

* The consideration of g and Γ as independent components of the field, in connection with other considerations, is also carried out in ⁽¹⁰⁾.

section of the A -th row and B -th column. It follows that the matrices e^{AB} possess the properties

$$(e^{AB})_{CD} = \delta_{AC}\delta_{BD}, \quad e^{AB}e^{CD} = \delta_{BC}e^{AD}. \quad (16)$$

We shall generalize this definition, taking as its basis the relations (16), but assuming that the quantities δ_{AB} are generalized Kronecker symbols (see (8)). In our case the indices A, B run through all values from the set $\{(ik), l(mn)\}$, and the corresponding quantities δ_{AB} are defined as follows:

$$\delta_{(ik),(i'k')} = \frac{1}{2}(\delta_{ii'}\delta_{kk'} + \delta_{ik'}\delta_{ki'}), \quad \delta_{l(ik),l'(i'k')} = \delta_{ll'}\delta_{(ik),(i'k')},$$

$$\delta_{(ik),l'(i'k')} = \delta_{l(ik),(i'k')} = 0, \quad (17)$$

where δ_{ik} are the ordinary Kronecker symbols. Here the basic role is played by the fact that the quantities δ_{AB} satisfy the relations $\delta_{AB} = \delta_{BA}$, $f_A\delta_{AB} = f_B$, characteristic of the ordinary Kronecker symbols. However, the matrices e^{AB} determined with their aid by conditions (16) may have several nonzero elements, and the latter are not necessarily equal to unity.

Using (16), (17), it is not hard to verify the validity of the following expressions for the matrices of equation (11):

$$\gamma^j = e^{(ik),j(ik)} - e^{(ij),k(ik)} + e^{j(ik),(ik)} - 2e^{i(jk),(ik)}, \quad (18)$$

$$\gamma^0 = e^{(ik),(ik)}, \quad \Lambda^{l(ik)} = e^{(lm),n(ik)} + e^{n(ik),(lm)}, \quad (19)$$

$$\Lambda^{(ik)} = e^{i(ik),n(ln)} - e^{n(il),l(kn)}. \quad (20)$$

The last matrix may be symmetrized (see (11)),

$$\Lambda^{(ik)} = \frac{1}{2}(e^{l(ik),n(ln)} + e^{n(ln),l(ik)} - e^{n(il),l(kn)} - e^{l(kn),n(il)}). \quad (21)$$

It is not hard to verify, with the aid of (16)–(18), that

$$\hat{p}^3(\hat{p}^2 - p^2) = 0, \quad \hat{p} = p_j\gamma^j, \quad p^2 = p_j^2, \quad (22)$$

where p_j are arbitrary numbers or mutually commuting operators. Relation (22) coincides with the minimal equation for the matrices of 30-dimensional equations describing a particle with spin 2 ^(7,9). For the matrices $\Lambda^{(ik)} = \beta$ the minimal equation has the form

$$\beta(\beta^2 - \frac{1}{2})(\beta^2 - 1) = 0 \quad \text{for } i = k; \quad \beta(\beta^2 - \frac{1}{2})(\beta^2 - \frac{1}{4}) = 0 \quad \text{for } i \neq k.$$

Einstein' s equations for the gravitational field in the presence of matter can also be represented in the form (11). For this purpose we replace in (8) λg_{ik} by R_{ik} and add to (8), (9) the equations

$$R - R_{ik}g^{ik} = 0, \quad g^{ik}g_{kl} = \delta_l^i, \quad R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik}. \quad (23)$$

This yields an equation of type (11),

$$(\gamma^j \partial_j + \gamma^0)\psi + \psi \Lambda \psi = \chi, \quad (24)$$

in which the functions ψ , χ , and, correspondingly, the matrix Λ will consist of five blocks,

$$\psi = (R_{ik}, \Gamma_{ik}^l, R, g^{ik}, g_{ik}), \quad \chi = (0, 0, 0, \delta_k^i, \kappa T_{ik}),$$

$$\Lambda = (\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)}, \Lambda^{(5)}). \quad (25)$$

In the corresponding block form the matrices of equation (24) have the form

$$\begin{aligned} \gamma^j &= \begin{pmatrix} 0 & \gamma_{R\Gamma}^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_{\Gamma g}^j \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_{\Gamma\Gamma}^R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda^{(2)} &= \begin{pmatrix} 0 & \Lambda_{R\Gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_{\Gamma g}^\Gamma \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (26) \\ \gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda^{(3)} = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\Lambda^{(4)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha^{(ik)} \\ 0 & 0 & 0 & \alpha^{(ik)} & 0 \end{pmatrix}, \quad \Lambda^{(5)} = -\frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta^{(ik)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\beta}^{(ik)} & 0 & 0 \end{pmatrix}.$$

Here the blocks of the 4 matrices γ^j , the 10 matrices $\Lambda^{(1)}$, and the 40 matrices $\Lambda^{(2)}$ are determined by the preceding formulas (18)–(21). The matrices γ^0 and $\Lambda^{(3)}$ include only unit matrices of the corresponding order, with, as before, $\gamma^{0^2} = \gamma^0$. In the matrices $\Lambda^{(4)}$, $\Lambda^{(5)}$ the blocks $\alpha^{(ik)}$ have the form $\frac{1}{2} (e^{(il),(lk)} + e^{(kl),(li)})$, while the blocks $\beta^{(ik)}$ are 10-dimensional row matrices having a unit in the place with number (ik) and zeros elsewhere. The total order of the matrices entering (24) reaches 71; however, in calculations this is of practically no importance, since expressions (26), (18)–(21) make it possible to do without writing out the matrices in expanded form.

Equations (11), (24) are of interest because their form makes it possible, to a certain extent, to use for the study of the gravitational field and, in particular, for its quantization, methods developed for relativistic wave equations describing elementary particles. In addition, they make it possible to look in a new way at the question of the exceptional character of the gravitational field in comparison with all other fields.

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