

THE DEFECT INDEX AND THE SPECTRUM OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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Abstract

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MATHEMATICS

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THE DEFECT INDEX AND THE SPECTRUM OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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1°. We shall consider self-adjoint operators generated in the Hilbert space of n -component vector-functions $L_n^2(a, b)$ by the differential expression:

$$l\bar{y} = \frac{i}{2}(\Lambda\bar{y})' + \frac{i}{2}\Lambda\bar{y}' + Q(x)\bar{y} \quad (a < x < b),$$

where $\Lambda(x)$, $Q(x)$ are Hermitian matrix-functions in the unitary space E_n , with $\Lambda(x)$ nondegenerate and absolutely continuous for $a < x < b$; $Q(x)$ is summable on every segment $[\alpha, \beta] \subset (a, b)$.

Let us agree on some notation: μA and νA are, respectively, the smallest and the largest eigenvalues of a Hermitian matrix A ;

$\operatorname{Re} A = (A + A^*)/2$, $\operatorname{Im} A = (A - A^*)/2i$.

Let L_0 be the minimal closed symmetric operator generated by the operation l , with domain of definition D_0 . In this section we shall be interested in the question of the defect index of the operator L_0 .

Theorem 1. Suppose that, for the operator L_0 , the endpoint a is regular, and in a neighborhood of the endpoint b the condition

$$\int_{x_0}^{\infty} \frac{dt}{\|\Lambda(t)\|} = \infty \quad (a < x_0 < b) \quad (1)$$

is satisfied.

Then the defect index of the operator L_0 has the form (ν_+, ν_-) , where ν_+ and ν_- are, respectively, the number of positive and the number of negative eigenvalues of the matrix $\Lambda(x)$ (counting their multiplicities).

Corollary 1.1. Suppose the endpoint b is regular, and in a neighborhood of $x = a$ the condition

$$\int_a^{x_0} \frac{dt}{\|\Lambda(t)\|} = \infty \quad (a < x_0 < b)$$

is satisfied.

Then the defect index of the operator L_0 has the form (ν_-, ν_+) .

Corollary 2.1. Suppose that in neighborhoods of the endpoints a and b the conditions

$$\int_{x_0}^b \frac{dt}{\|\Lambda(t)\|} = \infty, \quad \int_a^{x_0} \frac{dt}{\|\Lambda(t)\|} = \infty$$

are satisfied.

Then the operator L_0 is self-adjoint.

Corollary 3.1. Suppose the endpoint a is regular and condition (1) is satisfied. Then, if $\nu_- \neq \nu_+$, the core of the spectrum of the operator L_0 fills the entire real axis. In particular, the latter will always occur if n is odd.

2°. In the usual standard way one constructs the resolvents of self-adjoint extensions L of the operator L_0 (under the assumption that such extensions exist). In this case the resolvents turn out to be integral operators

with kernels of Carleman type:

$$R_\lambda \bar{f} = \int_a^b K(x, \xi, \lambda) \bar{f}(\xi) d\xi \quad (\text{Im } \lambda \neq 0);$$

$K(x, \xi, \lambda)$ is a matrix kernel of the form:

$$K(x, \xi, \lambda) = Y(x, \lambda) \left[M(\lambda) + \frac{1}{2} \text{sgn}(x - \xi) \Lambda_0^{-1} \right] Y^*(\xi, \bar{\lambda}), \quad (2)$$

where $Y(x, \lambda)$ is the matrix of the fundamental system of solutions of the homogeneous system $l\bar{y} = \lambda\bar{y}$, satisfying the condition $Y(a, \lambda) = E$; $M(\lambda)$ is the characteristic matrix-function of the operator L ; $\Lambda_0 = \Lambda(a)$. The relations

$$\int_a^b \|K(x, \xi, \lambda)\|^2 d\xi < \infty \quad (a < x < b),$$

$$\int_a^b \|K(x, \xi, \lambda)\|^2 dx < \infty \quad (a < \xi < b).$$

hold.

The case when the defect index of the operator L_0 is (n, n) will be called quasiregular. As is seen from the representation (2), in this case the kernel $K(x, \xi, \lambda)$ for $\text{Im } \lambda \neq 0$ is a Hilbert-Schmidt kernel; consequently, the spectrum of any self-adjoint extension L of the operator L_0 is discrete. Here we shall formulate some conditions for quasiregularity of the operator L_0 .

Theorem 2. Let the quadratic form $\Lambda[\bar{\xi}] = (\Lambda\bar{\xi}, \bar{\xi})$ be sign-definite and let the condition

$$\int_a^b \frac{dt}{|\lambda(t)|} < \infty,$$

be satisfied, where

$$\lambda(x) = \begin{cases} \Delta\Lambda(x), & \text{if } \Lambda[\bar{\xi}] \text{ is positive definite,} \\ \nabla\Lambda(x), & \text{if } \Lambda[\bar{\xi}] \text{ is negative definite.} \end{cases}$$

Then the operator L_0 is quasiregular.

In what follows we shall need the following estimates for solutions of a linear system:

Lemma 1. For any solution $\bar{y}(x)$ of the linear system $d\bar{Y}/dx = A(x)\bar{y}$, the estimates

$$\|\bar{y}(x)\|^2 \leq c_0 \exp \left[2 \int_{x_0}^x \nu \text{Re } A(\tau) d\tau \right],$$

$$\|\bar{y}(x)\|^2 \geq \varepsilon_0 \exp \left[2 \int_{x_0}^x \mu \text{Re } A(\tau) d\tau \right],$$

hold, where $c_0 = \|Y(x_0)\|^2$.

On the basis of Lemma 1 the following assertion is proved:

Theorem 3. Let the endpoint a be regular and let the following conditions be satisfied for the matrices $\Lambda(x)$ and $Q(x)$:

- 1) There exists an absolutely continuous function $\delta(x) > 0$ ($a < x < b$) such that: a)

$$\int_a^b \frac{dt}{\delta(t)} < \infty;$$

- b) the elements of the matrix

$$\Lambda_0(x) = \frac{1}{\delta(x)} \Lambda(x)$$

are functions whose moduli and arguments are monotone and bounded in a neighborhood of the endpoint b , and moreover

$$\Lambda_0(b) = \lim_x \Lambda_0(x)$$

is a nondegenerate matrix.

2)

$$\int_a^b \|\operatorname{Im}(\Lambda^{-1}(t)Q(t))\| dt < \infty.$$

Then the operator L_0 is quasiregular.

3°. Suppose that the endpoint a is regular and that condition (1) is satisfied for $\Lambda(x)$. We shall assume, moreover, that n is even ($n = 2m$).

By applying Rapoport's asymptotics for the solutions of the linear system $l\bar{y} = 0$ (see (1), p. 228), the following criterion for discreteness of the spectrum of the operator L is established:

Theorem 4. Let the following conditions be satisfied for the matrices $\Lambda(x)$ and $Q(x)$:

- 1) There exists an absolutely continuous function $\delta(x) > 0$ ($a \leq x < b$) such that

$$\int_{x_0}^b \frac{dt}{\delta(t)} = \infty \quad (a < x_0 < b),$$

and, for

$$\Lambda_0(x) = \frac{1}{\delta(x)} \Lambda(x),$$

condition 1b of Theorem 3 is satisfied.

- 2) The matrix $Q(x)$ is absolutely continuous, the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix $i\Lambda_0^{-1}Q$ are simple, and $\operatorname{Re} \lambda_i, \operatorname{Re}(\lambda_i - \lambda_k)$ ($1 \leq i \leq k \leq n$) do not change sign in a neighborhood of the endpoint b .
- 3) The matrix $B(x)$, whose columns are eigenvectors of the matrix $\Lambda_0^{-1}Q$, satisfies the condition: the moduli and arguments of the elements of the matrix $B(x)$ are monotone and bounded in a neighborhood of the endpoint b , and

$$B(b) = \lim_{x \rightarrow b} B(x)$$

is nonsingular.

Then, for the spectrum of any self-adjoint extension L of the operator L_0 to be discrete, it is necessary and sufficient that the following conditions be satisfied: for every $\omega > 0$,

$$\int_{\alpha}^{\beta} \frac{|\operatorname{Re} \lambda_i(t)|}{\delta(t)} d\tau \rightarrow \infty \quad (1 \leq i \leq n),$$

when $\alpha \rightarrow b$ and

$$\int_{\alpha}^{\beta} \frac{d\tau}{\delta(\tau)} = \omega.$$

4°. The following assertion holds, which is an analogue of a theorem of I. E. Shnol' (2), established by him for the Sturm-Liouville operator:

Theorem 5. Let the endpoint a be regular and

$$\int_{x_0}^b \frac{dt}{\|\Lambda(t)\|} = \infty \quad (a < x_0 < b).$$

Suppose that the system $l\bar{y} = \lambda\bar{y}$ has a solution $\bar{y}(x, \lambda)$ satisfying, for some sequences $\xi_p \rightarrow b$ and $\varepsilon_p \rightarrow 0$ as $p \rightarrow \infty$, one of the following estimates:

1)

$$\int_a^{\xi_p} \|\bar{y}(t, \lambda)\|^2 dt \leq c \exp \left[\varepsilon_p \int_a^{\xi_p} \frac{dt}{\|\Lambda(t)\|} \right],$$

if $\bar{y}(x, \lambda)$ does not belong to $L_n^2[a, b]$.

2)

$$\int_{\xi_p}^b \|\bar{y}(t, \lambda)\|^2 dt \geq c \exp \left[-\varepsilon_p \int_a^{\xi_p} \frac{dt}{\|\Lambda(t)\|} \right],$$

if $\bar{y}(x, \lambda)$ belongs to $L_n^2[a, b]$.

Then λ belongs to the continuous spectrum $C(L)$ of the operator L .

On the basis of Theorem 5 and the estimates of Lemma 1, the following assertion is proved:

Theorem 6. Let the endpoint a be regular and let condition 1) of Theorem 4 be satisfied for the matrix $\Lambda(x)$. Then, if

$$\liminf_{x \rightarrow b} \left\{ \int \|\operatorname{Im}(\Lambda^{-1}Q)\| dt / \int_a^x \frac{dt}{\delta(t)} \right\} = 0,$$

then the spectrum of the operator L fills the entire real axis.

5°. Consider the operator L , generated in $L_n^2(0, \infty)$ by a system of even order ($n = 2m$) of a more special form (the so-called Hamiltonian system):

$$l\bar{y} = I\bar{y}' + Q(x)\bar{y}, \quad \text{where } I = \begin{pmatrix} 0 & & & -1 \\ & \ddots & & \\ & & -1 & \\ 1 & & & \\ & \ddots & & \\ 1 & & & 0 \end{pmatrix}.$$

By q_i denote the diagonal elements of the matrix $Q(x)$: $q_i = q_{ii}$ ($1 \leq i \leq n$). Put $Q(x) = D(x) + Q_0(x)$, where $D(x)$ is the matrix on whose main diagonal stand the q_i ($1 \leq i \leq n$), and all remaining elements are zeros. Divide the matrix IQ_0 into 4 matrices of size $m \times m$:

$$IQ_0 = \begin{pmatrix} P_1 & M \\ K & P_2 \end{pmatrix}.$$

Denote

$$N = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

With the aid of Rapoport's asymptotics for solutions of the system $l\bar{y} = \lambda\bar{y}$ (see ⁽¹⁾, p. 232), the following assertion is proved:

Theorem 7. *Suppose that the following conditions are satisfied:*

- 1) For $m+1 \leq k \leq 2m$, $q_k(x)$ are monotone and $q_k(x) \rightarrow 0$.
- 2) For $1 \leq k \leq m$,

$$q_k(x) \xrightarrow{x \rightarrow \infty} \infty,$$

$q'_k(x)$ and $q''_k(x)$ preserve their sign for $x \geq x_0$, and

$$q'_k = O(|q|^\alpha) \quad \text{for } 0 < \alpha < \frac{3}{2}.$$

- 3) $q_i(x) - q_j(x) \geq 0$ for $1 \leq i \leq j \leq 2m$.
- 4) For $q_k(x)$, $1 \leq k \leq m$, there exists such a monotone function $q(x)$ ($q(x) \xrightarrow{x \rightarrow \infty} \infty$) that, for certain constants α_k and β_k ,

$$\alpha_k q(x) \leq q_k(x) \leq \beta_k q(x) \quad (x \geq x_0).$$

- 5) The elements of the matrices

$$\frac{1}{\sqrt{q}} K(x), \quad \sqrt{q} M(x) \quad \text{and} \quad N(x)$$

are summable on $[x_0, \infty)$.

Then the half-axis $\lambda > 0$ contains no points of the continuous spectrum of L .

Using Theorem 5, the following assertion is proved.

Theorem 8. Suppose that, in addition to the conditions of Theorem 7, the condition

$$\liminf_{x \rightarrow \infty} \frac{1}{x} \ln \int_{x_0}^x \sqrt{|q(s)|} ds = 0$$

is satisfied.

Then the continuous spectrum of the operator L fills the half-axis $\lambda \leq 0$, i.e.

$$C(L) = (-\infty, 0].$$

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Note: Figure translations are in progress. See original paper for figures.

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