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Abstract

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MATHEMATICS

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SOME CASES IN WHICH FOURIER SUMS GIVE AN APPROXIMATION OF THE ORDER OF THE BEST

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It is known that for functions of the class $W^r H_\omega$ (i.e., 2π -periodic functions having r -th derivatives, whose modulus of continuity $\omega(f^{(r)}, \delta)$ does not exceed the true majorant of moduli of continuity $\omega(\delta)$), the deviation of the partial sums of the trigonometric Fourier series has order $n^{-r}\omega(n^{-1}) \ln n$, and for the whole class $W^r H_\omega$ this order is definitive.

In the present note it is shown that for certain rather broad subclasses of the class $W^r H_\omega$ the factor $\ln n$ can be omitted. In other words, for functions from such subclasses the order of the deviation of Fourier sums coincides with the order of the quantity $\sup_{f \in W^r H_\omega} E_n(f)$, where $E_n(f)$ is the best approximation of the function f by polynomials

$$A_0 + \sum_{k=1}^n (A_k \cos kx + B_k \sin kx).$$

Lemma 1 (see ^(1,2)). If $f \in W^r H_\omega$ and $S_n[f; x]$ is the n -th partial sum of the trigonometric Fourier series of f , then

$$S_n[f; x] - f(x) = \frac{n^{1-r}}{2\pi^2} \sum_{k=1}^m \frac{1}{k} \int_{y_k}^{y_{k+1}} \left[f^{(r)} \left(x + \frac{r'\pi}{2n} + t \right) + f^{(r)} \left(x + \frac{r'\pi}{2n} - t \right) \right] \times \\ \times \sin nt \, dt + (r+1)O(n^{-r}\omega(n^{-1})),$$

where $m = (n-3)/2$, $y_k = (4k+1)\pi/2n$, $r' \in (-2, 2]$, $r' \equiv r \pmod{4}$; the constant entering the O -term is absolute.

Lemma 2 ⁽³⁾. If the majorant ω satisfies the condition

$$\delta \int_{\delta}^{\pi} \frac{\omega(t)}{t^2} dt = O(\omega(\delta)), \quad (*)$$

then there exist constants A and $\mu \in (0, 1)$ such that $\delta_1^{\mu} \omega(\delta_2) \leq A \delta_2^{\mu} \omega(\delta_1)$ for $0 < \delta_1 < \delta_2 \leq \pi$.

Theorem 1. Let $f \in H_{\omega} = W^0 H_{\omega}$ and be monotone on the intervals $[c_i, c_{i+1}]$, where $c_0 < c_1 < \dots < c_N = c_0 + 2\pi$. If ω satisfies condition $(*)$, then

$$S_n[f; x] - f(x) = O(\omega(n^{-1}) \ln(N+1)),$$

where the constant entering the O -term depends only on ω .

Proof. On the basis of Lemma 1 we have

$$\begin{aligned} S_n[f; x] - f(x) &= \frac{n}{2\pi^2} \left\{ \sum_{k=1}^m \frac{1}{k} \int_{y_k}^{y_{k+1}} f(x+t) \sin nt \, dt + \right. \\ &\quad \left. + \sum_{k=1}^m \frac{1}{k} \int_{y_k}^{y_{k+1}} f(x-t) \sin nt \, dt \right\} + O\left(\omega\left(\frac{1}{n}\right)\right). \end{aligned}$$

* $\omega(\delta)$ is a true majorant of moduli of continuity if: 1) ω is defined on $[0, +\infty)$;

2) $\lim_{\delta \rightarrow 0} \omega(\delta) = \omega(0) = 0$; 3) for $0 \leq \delta_1 < \delta_2$ one has $0 \leq \omega(\delta_2) - \omega(\delta_1) \leq \omega(\delta_2 - \delta_1)$.

Let $k_0, k_1, \dots, k_{p+1}$ be the numbers of those intervals $[y_k, y_{k+1}]$ into which at least one of the points $y_1, y_m, c_i - x$ falls (or a point differing from $c_i - x$ by an integral multiple of 2π). Obviously, $p \leq N$. Then (as usual, $\sum_{k=\mu}^{\nu} a_k = 0$ if $\mu > \nu$)

$$\begin{aligned} \Sigma_1 &= \sum_{k=1}^m \frac{1}{k} \int_{y_k}^{y_{k+1}} f(x+t) \sin nt \, dt = \sum_{i=0}^p \sum_{k=k_i+1}^{k_{i+1}-1} \frac{1}{k} \int_{y_k}^{y_{k+1}} f(x+t) \sin nt \, dt + \\ &\quad + \sum_{i=0}^{p+1} \frac{1}{k_i} \int_{y_{k_i}}^{y_{k_{i+1}}} f(x+t) \sin nt \, dt = \Sigma'_1 + \Sigma''_1. \end{aligned}$$

Since

$$\int_{y_k}^{y_{k+1}} f(x+t) \sin nt \, dt = O(n^{-1} \omega(n^{-1})),$$

we have

$$\Sigma_1'' = O\left(\frac{1}{n} \omega\left(\frac{1}{n}\right)\right) \sum_{i=0}^{p+1} \frac{1}{k_i} = O\left(\frac{1}{n} \omega\left(\frac{1}{n}\right)\right) \sum_{i=1}^{N+2} \frac{1}{i} = O\left(\frac{\ln(N+1)}{n} \omega\left(\frac{1}{n}\right)\right).$$

Since $f(x + \cdot)$ is monotone on $[y_{k_i+1}, y_{k_{i+1}}]$, for $k_i < k < k_{i+1}$ we have

$$\left| \int_{y_k}^{y_{k+1}} f(x+t) \sin nt \, dt \right| \leq \frac{2\pi}{n} |\Delta_{2\pi/n} f(x + y_k)|,$$

whence

$$|\Sigma_1'| \leq \frac{2\pi}{n} \sum_{i=0}^p \left| \sum_{k=k_i+1}^{k_{i+1}-1} \frac{1}{k} \Delta_{2\pi/n} f(x + y_k) \right|.$$

But

$$\begin{aligned} \sum_{k=k_i+1}^{k_{i+1}-1} \frac{1}{k} \Delta_{2\pi/n} f(x + y_k) &= \sum_{k=k_i+1}^{k_{i+1}-1} \frac{f(x + y_k) - f(x + y_{k_{i+1}})}{(k-1)k} + \\ &+ \frac{f(x + y_{k_{i+1}}) - f(x + y_{k_{i+1}})}{k_{i+1} - 1}, \end{aligned}$$

$$|f(x + y_k) - f(x + y_{k_{i+1}})| \leq \omega\left(\frac{2(k-1)}{n} \pi\right),$$

$$|f(x + y_{k_{i+1}}) - f(x + y_{k_{i+1}})| \leq \omega\left(\frac{2(k_{i+1} - 1 - k_i)}{n} \pi\right).$$

Thus,

$$\begin{aligned} |\Sigma_1'| &\leq \frac{4\pi}{n} \sum_{i=1}^p \left[\sum_{k=k_i+1}^{k_{i+1}-1} \frac{\omega((k-1)\pi n^{-1})}{(k-1)k} + \frac{\omega((k_{i+1} - 1 - k_i)\pi n^{-1})}{k_{i+1} - 1} \right] \leq \\ &\leq \frac{8\pi}{n} \sum_{k=2}^m \frac{\omega((k-1)\pi n^{-1})}{k^2} + \frac{4\pi}{n} \sum_{i=0}^p \frac{\omega((k_{i+1} - 1 - k_i)\pi n^{-1})}{k_{i+1} - 1}. \end{aligned}$$

The first of the resulting sums, by virtue of (*), is $O(n^{-1}\omega(n^{-1}))$.

To estimate the second sum we apply Lemma 2:

$$\frac{4\pi}{n} \sum_{i=0}^p \frac{\omega((k_{i+1}-1-k_i)\pi n^{-1})}{k_{i+1}-1} = O\left(\frac{1}{n} \omega\left(\frac{1}{n}\right)\right) \sum_{i=0}^p \frac{(k_{i+1}-1-k_i)^\mu}{k_{i+1}-1}.$$

Denote by l such a number that $k_l \leq (p+1)^{1/(1-\mu)} + 1 < k_{l+1}$ (if all the numbers $k_1, k_2, \dots, k_{p+1}$ lie on one side of $(p+1)^{1/(1-\mu)} + 1$, then

below one of the sums vanishes). Then

$$\begin{aligned} \sum_{i=0}^p \frac{(k_{i+1}-1-k_i)^\mu}{k_{i+1}-1} &\ll \sum_{i=0}^{l-1} \frac{k_{i+1}-1-k_i}{k_{i+1}-1} + \sum_{i=l}^p (k_{i+1}-1)^{\mu-1} < \\ &< \sum_{i=0}^{l-1} \sum_{k=k_i}^{k_{i+1}-1} \frac{1}{k} + \sum_{i=0}^p [(p+1)^{1/(1-\mu)}]^\mu < \frac{1}{1-\mu} \ln(p+1) + 2. \end{aligned}$$

Thus,

$$\Sigma'_1 = O\left(\frac{1}{n} \omega\left(\frac{1}{n}\right)\right) \left(1 + \frac{1}{1-\mu} \ln(p+1) + 2\right) = O\left(\frac{\ln(N+1)}{n} \omega\left(\frac{1}{n}\right)\right).$$

Consequently,

$$\Sigma_1 = O(\ln(N+1)\omega n^{-1}/n).$$

Similarly,

$$\Sigma_2 = \sum_{k=1}^m \frac{1}{k} \int_{y_k}^{y_{k+1}} f(x-t) \sin nt \, dt = O\left(\frac{\ln(N+1)}{n} \omega\left(\frac{1}{n}\right)\right).$$

The theorem is proved.

Remark 1. If $\omega(\delta) = \delta^\mu$, where $0 < \mu < 1$, then

$$S_n[f; x] - f(x) = O\left(\frac{\ln(N+1)}{(1-\mu)n^\mu}\right),$$

and the constant entering the O -term is absolute.

Remark 2. The condition of piecewise monotonicity of f cannot be replaced by the condition of bounded variation.

Remark 3. The condition $(*)$ imposed on the majorant ω is essential.

Lemma 3. If f is convex upward on $[y_k, y_{k+1}]$, then

$$\frac{4}{\pi n} \Delta_{\pi/n}^2 f(y_k) \ll \int_{y_k}^{y_{k+1}} f(t) \sin nt \, dt \ll 0.$$

Proof. We shall show that

$$-\frac{8}{\pi} g(\pi) \ll \int_0^{2\pi} g(z) \cos z \, dz \ll 0,$$

if g is convex upward on $[0, 2\pi]$, $g(0) = g(2\pi) = 0$. Let

$$h(z) = [g(z) + g(2\pi - z)]/2, \quad l(z) = 2h(\pi/2)z/\pi.$$

Then

$$\begin{aligned} \int_0^{2\pi} g(z) \cos z \, dz &= 2 \int_0^{\pi} h(z) \cos z \, dz \gg \\ &\gg 2 \int_0^{\pi} l(z) \cos z \, dz = -\frac{8}{\pi} h\left(\frac{\pi}{2}\right) \gg -\frac{8}{\pi} g(\pi). \end{aligned}$$

The inequality

$$\int_0^{2\pi} g(z) \cos z \, dz \ll 0$$

is obvious. To complete the proof it is enough to put

$$f\left(\frac{z}{n} + y_k\right) - \frac{z}{2\pi} [f(y_{k+1}) - f(y_k)] - f(y_k) = g(z)$$

and to note that

$$g(\pi) = -\frac{1}{2} \Delta_{\pi/n}^2 f(y_k).$$

Theorem 2. If $f \in H_\omega$ and is convex (upward or downward) on each of the intervals $[c_i, c_{i+1}]$, where $c_0 < c_1 < \dots < c_N = c_0 + 2\pi$, then

$$S_n[f; x] - f(x) = O(\omega(n^{-1}) \ln(N+1)),$$

where the constant entering the O -term is absolute.

Proof. With the preceding notation it is enough to show that

$$\Sigma'_1 = O(\ln(N+1)\omega n^{-1}/n).$$

The function $f(x + \cdot)$ preserves the direction of convexity on

$$[y_{k_i+1}, y_{k_{i+1}}].$$

Hence, by Lemma 3,

$$|\Sigma'_1| \ll \sum_{i=0}^p \left| \sum_{k=k_i+1}^{k_{i+1}-1} \frac{1}{k} \int_{y_k}^{y_{k+1}} f(x+t) \sin nt \, dt \right| \ll \frac{4}{\pi n} \sum_{i=0}^p \left| \sum_{k=k_i+1}^{k_{i+1}-1} \frac{1}{k} \Delta_{\pi/n}^2 f(y_k) \right|.$$

Moreover, $y_{k+1/2} = (4k+3)\pi/2n$,

$$|\Sigma'_1| \leq \frac{8}{\pi n} \sum_{i=0}^p \left| \sum_{\nu=2k_i+2}^{2k_{i+1}-2} \frac{1}{\nu} \Delta_{\pi/n}^2 f(y_{\nu/2}) \right|.$$

Next

$$\sum_{\nu=2k_i+2}^{2k_{i+1}-2} \frac{1}{\nu} \Delta_{\pi/n}^2 f(y_{\nu/2}) = \sum_{\nu=2k_i+2}^{2k_{i+1}-2} \frac{\Delta_{\pi/n} f(y_{\nu/2})}{(\nu-1)\nu} - \frac{\Delta_{\pi/n} f(y_{k_i+1})}{2k_i+1} + \frac{\Delta_{\pi/n} f(y_{k_{i+1}-1/2})}{2k_{i+1}-2}$$

and $|\Delta_{\pi/n} f(x)| \leq \omega(\pi/n)$. Therefore

$$\left| \sum_{\nu=2k_i+2}^{2k_{i+1}-2} \frac{1}{\nu} \Delta_{\pi/n}^2 f(y_{\nu/2}) \right| \leq \omega\left(\frac{\pi}{n}\right) \left[\sum_{\nu=2k_i+1}^{2k_{i+1}} \frac{1}{(\nu-1)\nu} + \frac{1}{k_i} \right].$$

Thus,

$$|\Sigma'_1| \leq \frac{8}{\pi n} \omega\left(\frac{\pi}{n}\right) \left[\sum_{i=0}^p \sum_{\nu=2k_i+1}^{2k_{i+1}} \frac{1}{(\nu-1)\nu} + \sum_{i=0}^p \frac{1}{k_i} \right] = O\left(\frac{\ln(N+1)}{n} \omega\left(\frac{1}{n}\right)\right),$$

which was required to be proved.

We shall call a function f q -monotone on $[a, b]$ if, for all x and $h > 0$ such that $x, x + qh \in [a, b]$,

$$\Delta_h^q f(x) = \sum_{\nu=0}^q (-1)^{q+\nu} C_q^\nu f(x + \nu h) \geq 0 \ (\leq 0).$$

It is not difficult to verify that: a) if f is q -monotone on $[a, b]$ and f' exists, then f' is $(q-1)$ -monotone on $[a, b]$; b) if a continuous f is q -monotone on $[a, b]$, then either f is $(q-1)$ -monotone on $[a, b]$, or there exists a $c \in (a, b)$ such that f is $(q-1)$ -monotone on $[a, c]$ and $[c, b]$.

Theorem 3. Let $f \in WrH_\omega$ and let f be q -monotone on the intervals $[c_i, c_{i+1}]$, where $c_0 < c_1 < \dots < c_N = c_0 + 2\pi$. Suppose, further, that $r \leq q-1$, and for $r = q-1$ condition (*) is satisfied for ω . Then

$$S_n[f; x] - f(x) = O((q + \ln(N+1))\omega(n^{-1})n^{-r}),$$

where the constant entering the O -term is absolute for $r < q-1$ and depends only on ω for $r = q-1$.

Proof. For $r = q-1$ the function $f^{(r)}$ is monotone on the intervals $[c_i, c_{i+1}]$. The proof of Theorem 1 (with insignificant modifications) gives the required result. If $r < q-1$, then $f^{(r)}$, being $(q-r)$ -monotone on $[c_i, c_{i+1}]$, is convex on the intervals $[d_i, d_{i+1}]$, where $d_0 < d_1 < \dots < d_M = d_0 + 2\pi$ and $M \leq 2^{q-r-2}N$. Slightly modifying the proof of Theorem 2, here also we obtain the needed estimate.

Assertions analogous to those proved are apparently also valid for Fourier series in Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$. We confine ourselves to formulating the following result obtained by us.

Let $0 < \mu \leq 1$, $-1/2 < \alpha, \beta < 1/2$, and let $S_n^{(\alpha, \beta)}[f; x]$ be the n -th partial sum of the Fourier-Jacobi series of a function f defined on $[-1, 1]$. Then, as shown in ⁽⁴⁾,

$$\sup_{f \in \text{Lip } 1\mu} |S_n^{(\alpha, \beta)}[f; 1] - f(1)| \asymp n^{\alpha+1/2-\mu}.$$

If, however, $f \in \text{Lip } 1\mu$ and the function $f(\cos \cdot)$ is convex on the intervals $[c_i, c_{i+1}]$, where $0 = c_0 < c_1 < \dots < c_N = \pi$, then

$$S_n^{(\alpha, \beta)}[f; 1] - f(1) = O(Nn^{\alpha-1/2-\mu} + N^{\alpha+1/2+\mu}n^{-2\mu} + n^{-\mu}).$$

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¹ A. V. Efimov, *Izv. AN SSSR, ser. matem.*, **24**, 243 (1960).

² G. I. Natanson, *Vestn. LGU*, No. 19, 20 (1966).

³ N. K. Bari, S. B. Stechkin, *Tr. Mosk. matem. obshch.*, **5**, 483 (1956).

⁴ S. A. Atakhanov, G. I. Natanson, *DAN*, **166**, 9 (1966).

Note: Figure translations are in progress. See original paper for figures.

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