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Abstract

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MATHEMATICS

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ON THE TOPOLOGICAL THEORY OF CONDENSING OPERATORS

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In the note ⁽¹⁾, B. N. Sadovskii drew attention to a new interesting class of nonlinear operators, which he called condensing, and proved for them an analogue of Schauder's fixed-point theorem.

In the present note we develop a method that makes it possible to investigate more general problems for this class of operators. The basic idea is that a condensing operator can be narrowed down to a compact convex subspace, relative to which the rotation of the corresponding vector field, studied earlier by one of the authors ^(4,5), is defined. This permits one to prove for condensing operators a number of assertions having analogues in the theory of completely continuous and weakly continuous vector fields ^(2,3).

1. In this section we study an important property of condensing operators—the property of having invariant compact subspaces.

Let us recall the basic definition ⁽¹⁾. The **measure of noncompactness** $\chi(B)$ of a set B is the infimum of those ε for which the set B has a finite ε -net. An operator $F : \Omega \rightarrow E$ is called **condensing** on the set Ω if $\chi(FB) < \chi(B)$ for every bounded noncompact subset $B \subset \Omega$. The equality $\chi(B) = \chi(\overline{\text{co}}B)$ holds, where $\overline{\text{co}}B$ is the convex closure of the set B .

Let now $F : \overline{U} \rightarrow E$ be a condensing mapping defined on the closure of a bounded open set $U \subset E$.

Theorem 1. *For any compact set R , $R \cap U \neq \emptyset$, there exists a compact closed convex set A such that:*

1) $R \subset A$, 2) $U_A = U \cap A \neq \emptyset$, 3) *the restriction of the mapping F to $\overline{U}_A$ acts in A : $F\overline{U}_A \subset A$.*

Proof. Consider the class $\{B\}$ of all bounded convex closed sets satisfying the conditions: a) $R \subset B$, b) $F\overline{U}_B \subset B$. We shall call the sets B admissible. The class of admissible sets is nonempty; for example, $B = \overline{\text{co}}(R \cup FU)$ is admissible. Let $A = \bigcap B$. This is the minimal set in the class $\{B\}$, since a) is fulfilled and $F\overline{U}_A \subset F\overline{U}_B \subset B$ for every B .

Consider the operator $PB = \overline{\text{co}}(R \cup FU_B)$, which assigns to sets from $\{B\}$ the convex closed sets PB . We have the inclusions $PB \subset B$, $PB \subset \{B\}$. The first follows from conditions a), b), and the second from the inclusions $R \subset PB$, $F\overline{U}_{PB} \subset F\overline{U}_B \subset PB$. In particular, $PA \subset A$, which, by the minimality of A , leads to $PA = A$. The latter is possible only for compact A , since otherwise, when $\chi(A) > 0$, we have the contradictory inequality

$$\chi(A) = \chi(PA) = \chi(F\overline{U}_A) < \chi(A).$$

Thus theorem 1 is proved.

Let now $K \subset E$ be a cone in the Banach space E , and let $U_K = U \cap K$ be a bounded open set relative to K . Let $F : \overline{U}_K \rightarrow K$ be a condensing positive mapping. By $K_\theta = \theta[A]$ we shall denote the cone formed by all rays issuing from the point θ and intersecting the convex set $A \subset E$. It is obvious that if A is compact and separated from zero, then the cone $\theta[A]$ is locally compact. We shall show that a positive condensing mapping, under the condition $\theta \in F\overline{U}_K$

induces, in a certain way, a mapping $\tilde{F}$ acting in a locally compact cone and closely connected with F ; we shall call $\tilde{F}$ the restriction of F .

Theorem 2. For every compact set $R \subset K$, $\theta \in R$, $R \cap U \neq \emptyset$, there exists a compact closed convex set $A \subset K$, $\theta \in A$, such that: 1) $R \subset A$; 2) $U \cap K_0 \neq \emptyset$, where $K_0 = \theta[A]$; 3) the restriction of the mapping F to $\overline{U}_{K_0}$ acts in K_0 : $\tilde{F}\overline{U}_{K_0} \subset K_0$.

Proof. Consider the class $\{D\}$ of all sets satisfying the conditions*: a) $D = [\theta B]$; b) $B \subset K$ is a convex closed bounded set; c) $R \subset B$; d) $F\overline{U}_D \subset B$; e) $\theta \in B$.

As in the first theorem, it is easy to verify that the class $\{D\}$ contains a minimal set $D_0 = \bigcap D$, with $D_0 = [\theta A]$, where $A = \bigcap B$. It remains to show that A is compact. To this end consider the operator in the space of subsets

$$PD = [\theta(R \cup F\overline{U}_D)].$$

Analogously, we have $PD \subset D$, $PD \subset \{D\}$, whence it follows that $PD_0 = D_0$ and, consequently, $\chi(F\overline{U}_{D_0}) = \chi(A)$. The latter implies the compactness of A , since for $\chi(A) > 0$ we have $\chi(F\overline{U}_{D_0}) < \chi(A)$. Extending F from $\overline{U}_{D_0}$ to $\overline{U}_{K_0}$ by constant values, we obtain $\tilde{F}$, which proves the theorem.

Suppose now that the condensing operator F_t depends on a parameter $t \in [0, 1]$ in such a way that the condensing property is preserved for every t and $F_t x$ depends continuously on t uniformly in x on $\overline{U}$ or $\overline{U}_K$; let $\theta \in F_t \overline{U}_K$, $0 \leq t \leq 1$. We shall call F_t a family of homotopic condensing operators.

Theorem 3. Let F_t be a family of homotopic condensing mappings $\overline{U} \rightarrow E$ or $\overline{U}_K \rightarrow K$. Then the assertions of Theorems 1 and 2 are valid with the modification that: 3) $F_t \overline{U}_A \subset A$, $\tilde{F}_t \overline{U}_{K_0} \subset K_0$ for all $t \in [0, 1]$.

In the proof we use new properties of the function $\chi(B)$. Let

$$\theta(A, B) = \sup_{a \in A} \rho(a, B)$$

be the deviation of the set A from the set B , and

$$\rho(A, B) = \max\{\theta(A, B), \theta(B, A)\}$$

the distance between A and B in the sense of Hausdorff.

Lemma 1. $\chi(B) = \inf \theta(B, \sigma)$, where σ is an arbitrary finite subset of E .

Lemma 2. If G_t is a family of bounded sets, continuously dependent on $t \in [0, 1]$ in the Hausdorff metric, then $\chi(G_t)$ is a continuous function of t and

$$\chi\left(\bigcup_t G_t\right) = \sup_t \chi(G_t) = \chi(G_{t_0})$$

for some $t_0 \in [0, 1]$.

We now indicate what changes must be made in the proofs of Theorems 1, 2 in order to obtain Theorem 3. Suppose first that $F_t : \bar{U} \rightarrow E$. Define the class of admissible sets $\{B\}$ by the conditions: a) $R \subset B$; b) $F_t \bar{U}_B \subset B$ for every t . This class includes, for example, the set $\text{co}(R \cup \bigcup_t F_t U)$. Consider the operator

$$PB = \overline{\text{co}}\left(R \cup \bigcup_t F_t U_B\right),$$

for which property 3) is satisfied.

The minimal set A in the class $\{B\}$ satisfies the condition $PA = A$, which is equivalent to the compactness of A and $F_t \bar{U}_A \subset A$ for all $t \in [0, 1]$. Here we use the inequality

$$\chi\left(\bigcup_t F_t \bar{U}_A\right) < \chi(A)$$

for $\chi(A) > 0$, proved with the aid of Lemma 2. The proof is carried out analogously in the case of positive mappings F_t .

2. In this section we shall define the relative rotation of the vector field generated by a condensing operator.

Let $F_t : \bar{U} \rightarrow E$ or $\bar{U}_K \rightarrow K$ be a homotopic family of condensing operators, and suppose that on the boundary $\dot{U}$ or $\dot{U}_K$ ($\dot{U}_K$ is understood relative to K) there are no fixed points of the operator F_t , $t \in [0, 1]$. It is easy to see that the set G of all fixed points of the operators F_t is compact. Choose a compact set R so that $R \supset G$, $U \cap R \neq \emptyset$, and so that the relative boundary $\dot{U}_R$ is nonempty. Then construct for the family

* We denote $\overline{\text{co}}(\theta \cup B)$ by $[\theta B]$.

F_t has a common invariant subspace A or K_0 , as in Theorem 3, and we narrow the family F_t to this subspace: $F_t : \bar{U}_A \rightarrow A$, $F_t : \bar{U}_{K_0} \rightarrow K_0$. On the relative boundaries $\dot{U}_A, \dot{U}_{K_0}$ the operators F_t have no fixed points. For the corresponding vector fields $\Phi_t x = F_t x - x$, relative to A or K_0 , on the boundaries $\dot{U}_A, \dot{U}_{K_0}$ there is defined the rotation $\gamma(\dot{U}_A, \Phi_t)$, $\gamma(\dot{U}_{K_0}, \Phi_t)$, studied earlier by one of the authors ⁽⁴⁾. It is constant under changes of t and has the usual properties.

If one analyzes a number of topological theorems ⁽²⁻⁴⁾, one observes that all constructions can be carried out in an invariant convex compact subset. This makes it possible, with the aid of the relative rotation $\gamma(\dot{U}_A, \Phi_t)$, $\gamma(\dot{U}_{K_0}, \Phi_t)$, to obtain analogous propositions for condensing operators.

3. We formulate a number of such assertions. In notes ^(4,5) one of the authors considered various strengthenings of Brouwer's theorem ⁽⁶⁾. We shall formulate their analogues and a number of other propositions for condensing mappings.

Theorem 4. Let $F : \bar{U} \rightarrow E$ be a condensing mapping, $D \subset U$ some convex closed set, $F^{jD} \subset U$ for all $j = 1, 2, \dots, m-1$, and let F^m map its natural domain of definition into D . Then the relative rotation of the vector field $\Phi = F - I$ is equal to 1; consequently, the mapping F has a fixed point in D .

Theorem 5. Let U be an open convex set. Let $F, F^1, \dots, F^{2m-1}$ be condensing mappings $\bar{U} \rightarrow E$, with $F^k(\bar{U}) \subset U$, $k = m, \dots, 2m-1$. Then the operator F has a fixed point.

We note that in this case the relative rotation is also equal to 1.

Theorem 6. Let F_t , $0 \leq t \leq 1 : \bar{U} \rightarrow E$ be a family of homotopic condensing mappings, where F_0 satisfies the conditions of Theorem 4 or 5, and on the boundary $\dot{U}$ there are no fixed points of any of the operators F_t . Then the operator F_1 has a fixed point, and the rotation of the vector field $\Phi_1 = F_1 - I$ is equal to +1.

Corollary. If F_0 is a condensing mapping $\bar{U} \rightarrow E$ satisfying the conditions of Theorems 4 or 5, and if the operator $F_0 + \Omega : \bar{U} \rightarrow E$ is condensing, then there exists $\varepsilon > 0$ such that the mapping $F_0 + \mu\Omega$, for all μ , $0 \leq \mu \leq \varepsilon$, has a fixed point.

Theorem 7. If F is a condensing mapping $\bar{U} \rightarrow E$ in a Hilbert space, $\theta \in U$, and $(Fx, x) \leq (x, x)$, $x \in \dot{U}$, then there exists a fixed point.

Theorem 8. If F is a condensing mapping, defined in a neighborhood of the point θ , having a Fréchet derivative F'_θ at the point θ , which is also a condensing operator, then every eigenvalue λ , $|\lambda| > 1$, of the Fréchet derivative of odd multiplicity is a bifurcation point of small solutions of the equation $Fx = \lambda x$.

Now let $F : \bar{U}_K \rightarrow K$ be a positive condensing operator. We note the simple proposition:

If $U \ni \theta$ and on the boundary $\dot{U}_K$ the inequality $Fx \geq x$ holds (“not a single element goes forward” ⁽³⁾), then $\gamma(\Phi, \dot{U}_{K_0}) = 1$.

The results of the article are valid for countably normed spaces, if one sets $\chi = \sup \chi_\alpha$, where χ_α is the measure of noncompactness constructed from the seminorm $\|x\|_\alpha$.

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REFERENCES

1. B. N. Sadovskii, *Functional Analysis and Its Applications*, 1, no. 2, 1967, p. 74.
2. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
3. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, 1962.
4. Yu. G. Borisovich, DAN, 153, no. 1, 12 (1963).
5. Yu. G. Borisovich, *Transactions of the Seminar on Functional Analysis, Voronezh State University*, issue 7, 17 (1963).
6. F. E. Browder, *Duke Math. J.*, 26, no. 2 (1959).

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