

ON PROBLEMS OF MATHEMATICAL PROGRAMMING HAVING A MINIMAL POINT

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Abstract

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MATHEMATICS

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ON PROBLEMS OF MATHEMATICAL PROGRAMMING HAVING A MINIMAL POINT

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In this note it is shown that certain problems of mathematical (linear and non-linear) programming in a partially ordered Banach space are equivalent to the problem of fixed points of operators leaving invariant a nonnegative cone. Questions concerning the existence of such points are set forth in detail in the monograph of M. A. Krasnosel'skii ⁽¹⁾, from which we borrow the notation and terminology. In the problems considered here, the solution can always be obtained by the method of successive approximations.

Let E be a Banach space partially ordered by the relation $\geq$ with the aid of a cone K . We shall assume the cone K to be regular and minihedral. Typical examples may be taken to be the spaces L_p and finite-dimensional Euclidean space with their natural nonnegative cones.

Let $f : K \rightarrow E$ be a continuous monotone operator (if $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$) and

$$D = \{x \mid x \geq f(x), x \geq \theta\}, \quad (1)$$

Consider the problem of mathematical programming

$$\varphi(x) = \min, \quad x \in D, \quad (2)$$

where φ is a monotone functional defined on K . For the finite-dimensional case, as early as 1962 E. B. Ershov ⁽²⁾ observed that if the set D is nonempty, then it contains a minimal point

$$z = \inf D,$$

which, naturally, will be a solution of problem (2), independently of the specific choice of the monotone functional φ . But Ershov did not know an algorithm for finding this point.

Let

$$\begin{aligned} F(x) &= \sup(\theta, f(x)), & x \in K; \\ N &= \{x \mid x = F(x)\}. \end{aligned} \quad (3)$$

The operator $F : K \rightarrow K$ thus defined is continuous and monotone together with f . The continuity of F follows from the regularity of the cone K .

Theorem 1. The following assertions are equivalent:

- 1) D is nonempty;
- 2) N is nonempty;
- 3) there exists the limit z of the sequence

$$z_0 = \theta, \quad z_{k+1} = F(z_k), \quad (4)$$

and $z \in N \subset D$ and $z = \inf D (= \inf N)$.

Proof. From 2) follows 1), since $N \subset D$. From 3) follows 2) by the continuity of F . It remains to show that from 1) follows 3). Let

$y \in D$. Then for the operator F we have

$$F(\theta) \geq \theta, \quad \theta \leq F(y) \leq y,$$

and, by virtue of the known theorem on the convergence of successive approximations (see ^(1,3)), the limit z of the sequence (4) exists, with $z \in N \subset D$ and $z \leq y$. Since y is an arbitrary element of D , it follows that $z = \inf D$. The proof is complete.

Thus, the convergence of the sequence (4) is a necessary and sufficient condition for the solvability of problem (2), and the limit of this sequence gives the minimal point. This aspect is most important from the computational point of view in solving practical finite-dimensional problems. In some applications (for example in economics, see ⁽⁴⁾) the operator f and the functional φ depend increasingly on a numerical parameter λ , and it is necessary to find the maximum value of λ for which problem (2) is solvable. Theorem 1 makes it possible to solve this problem as well, again using the sequence (4). In this sense the sequence (4) provides a decision algorithm for problem (2).

It turns out that this algorithm can also be used in the case when the operator f is not monotone. We shall demonstrate this by the example of a linear operator, although the same can also be done in nonlinear cases.

Let

$$f(x) := Ax + b, \quad A = P - S, \quad (5)$$

where P and S are linear positive operators and the spectral radius of the operator P is less than one. Then on the whole space E one can define the operator

$$T(y) = \inf\{x \mid x \geq Px - Sy + b, \ x \geq \theta\}, \quad (6)$$

and the image of the cone K is bounded,

$$\theta \leq T(K) \leq T(\theta).$$

The estimate holds

$$|T(y_1) - T(y_2)| \leq B|y_1 - y_2|, \quad B = (E - P)^{-1}S, \quad (7)$$

where

$$|x| = \sup(\theta, x) + \sup(\theta, -x).$$

We now define a monotone operator $G : K \rightarrow K$

$$G = T^2. \quad (8)$$

Since

$$T(\theta) \geq GT(\theta),$$

then, on the basis of Theorem 1, G has a nonempty set of fixed points, the least of which z can be obtained as the limit of the sequence (4) (with $F = G$), and the greatest w (which also exists) is determined as $w = T(z)$ (note that $z = T(w)$). Using this construction, the following can be proved.

Theorem 2. Let P and S be positive linear operators; let the operators A, f, F, T, B , and G be defined by equalities (5), (3), (6), (7), and (8), and let the domain D be defined by equality (1). Then, if the spectral radius of the matrix $(P + S)$ is less than one, then:

- 1) the operators F, T , and G have a unique fixed point z , common to them, which can be obtained as the (common) limit of the sequences $z_{k+1} = F(z_k)$ and $y_{k+1} = T(y_k)$ for arbitrary initial points z_0 and y_0 ;
- 2) the cone

$$M = \{c \mid (c, x) \geq 0, (c, (E - B)x) \geq 0, \text{ for all } x \in K\}$$

in the conjugate space of linear functionals on E is nonempty;

5) the point z minimizes on the set D any linear functional $c \in M$.

Property 3) of the point z allows us to call it quasiminimal. If we consider the spectrum of problems with the matrix

$$A_\lambda = P - \lambda S, \quad 0 \leq \lambda \leq 1,$$

then for all λ Theorem 2 remains valid, and the point z_λ , which for $\lambda = 0$ is simply a minimal point (in this case $B = 0$), for $\lambda = 1$ passes into the point z . The cone of minimized functionals under this homotopy narrows.

Let us also note that if the operator A is given, then its representation in the form

$$A = P - S$$

is not unique. In cases where A is a regular operator (see (4)), the best decomposition will be

$$P = A^+ = (|A| + A)/2, \quad S = A^- = (|A| - A)/2.$$

In this case

$$P + S = |A|.$$

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Note: Figure translations are in progress. See original paper for figures.

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