

A GENERALIZATION OF LYAPUNOV' S FIRST STABILITY THEOREM FOR CERTAIN CLASSES OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

A. E. Gel'man, E. N. Gersht

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.62475>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract**Full Text**

UDC 517.919.2:517.948

MATHEMATICS

A. E. GELMAN, E. N. GERSHT

**A GENERALIZATION OF LYAPUNOV'S
FIRST STABILITY THEOREM FOR CERTAIN
CLASSES OF DIFFERENTIAL EQUATIONS
IN BANACH SPACES***(Presented by Academician V. I. Smirnov on 20 IV 1967)*

§ 1. We consider the differential equation

$$\dot{z} = Az + f(t, z) \quad (1)$$

with the initial condition

$$z(0) = z_0. \quad (2)$$

Here $z = z(t)$ is the sought continuous, continuously differentiable on the half-axis $[0, \infty)$, vector-valued function with values in an arbitrary complex Banach space Z ; A is a linear operator with values in Z , with domain $D(A)$, dense in Z , and with spectrum in the left half-plane, generally speaking unbounded; $f(t, z)$ is a function defined and continuous on $[0, \infty) \times S$, where $S = \{z; z \in Z; \|z\| < \rho\}$, with values in Z , bounded with respect to t and, for each t , analytic in z , and its F -series has the form*

$$f(t, z) = \sum_{k=2}^{\infty} P_k^{(f)}(t)(z); \quad (3)$$

 z_0 is a fixed element of Z .**

The main results of the present paper are stated in Theorems 1 and 2 and, briefly, are as follows:

1. The trivial solution of equation (1) is asymptotically stable.
2. The domain of asymptotic stability contains some sphere $\|z_0\| < \Lambda$, whose radius Λ is effectively determined.

3. The solution of equation (1) satisfying the initial condition (2) is an analytic function of the vector of initial conditions in the sphere $\|z_0\| < \Lambda$ and an analytic function of the solution of the corresponding linear problem

$$\dot{z} = Az; \quad z(0) = z_0 \quad (4)$$

in the same sphere.

These results are a generalization of Lyapunov's first theorem ⁽¹⁾ for the case when the operator A is bounded and the function $f(t, z)$ is nonanalytic; questions of stability of the trivial solution of equation (1) were investigated by M. G. Krein ⁽²⁾. In the present paper the results of ⁽³⁾ are essentially used.

§ 2. Following ⁽³⁾, we shall say that a function of a complex argument x , holomorphic in a neighborhood of the point $x = 0$,

$$y(x) = \sum_{i=0}^{\infty} a_i x^i \quad (a_i \geq 0)$$

* The notation of ⁽⁴⁾ is used in the paper.

** The exact formulation of the problem is given in § 2.

majorizes the analytic function $\psi(z)$ with values in the complex B -space U , defined in a neighborhood of the point Θ_Z of the complex B -space Z ,

$$\psi(z) = \sum_{i=0}^{\infty} P_i^{(\psi)}(z),$$

if $\|P_i^{(\psi)}\| \leq a_i$.

Let $\Omega(z, \varphi)$ be an analytic function of two variables in some neighborhood of the point (Θ_Z, Θ_Φ) of the space $Z \times \Phi$, with values in the space U (Z, Φ, U are complex Banach spaces):

$$\Omega(z, \varphi) = \sum_{i,j=0}^{\infty} P_{ij}^{(\Omega)}(z, \varphi). \quad (5)$$

We introduce the notation*

$$\|P_{ij}^{(\Omega)*}\| = \sup \|P_{ij}^{(\Omega)*}(z_1, z_2, \dots, z_i, \varphi)\|, \quad \|z_k\| = \|\varphi\| = 1 \quad (k = 1, 2, \dots, i).$$

In what follows we shall assume that equation (1) satisfies conditions I or conditions II.

Conditions I.

Ia) The operator A belongs to the B -algebra of endomorphisms of the space Z and has spectrum $\sigma(A)$ lying in the left half-plane $\operatorname{Re} \lambda < 0$.

Ib) The power series

$$\bar{f}(x) = \sum_{k=2}^{\infty} \sup_{t \in [0, \infty)} \|P_k^{(f)*}(t)\| x^k \quad (6)$$

has a radius of convergence different from zero.

As is known from ^(2,4), in this case the operator A is the infinitesimal generator of a uniformly continuous semigroup of linear bounded operators $T(t) = e^{At}$, and for any positive

$$\nu < \inf_{\lambda \in \sigma(A)} |\operatorname{Re} \lambda|$$

there exists a constant $N > 0$ such that

$$\|T(t)\| \leq N e^{-\nu t} \quad (t \geq 0). \quad (7)$$

Conditions II.

IIa) A is an unbounded operator, a linear operator with domain of definition $D(A) \in Z$, with values in Z ; the operator A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ of class C_0 , of negative type, of linear bounded transformations.

IIb) The function $f(t, z)$ does not depend on t .

As is known ⁽⁴⁾, an operator A satisfying condition IIa) is closed and $D(A)$ is dense in Z . If $\omega_0 < 0$ is the type of the semigroup $T(t)$, then the spectrum of the operator A lies in the half-plane $\operatorname{Re} \lambda \leq \omega_0$, and for any positive $\nu < |\omega_0|$ there exists a constant $N > 0$ such that, for $\|T(t)\|$, inequality (7) holds.

Broad and important classes of operators satisfying condition IIa) are known (see, for example, ^(5,6)).

§ 3. Consider the equation

$$z = T(t)z_0 + \int_0^t T(t-s)f[s, z(s)] ds, \quad (8)$$

where $T(t)$ is a semigroup, strongly continuous for $t \geq 0$, of linear bounded operators satisfying inequality (7), and $f(t, z)$ satisfies condition Ib).

* $P_{ij}^{(\Omega)*}(z_1, z_2, \dots, z_i, \varphi)$ is the multilinear symmetric form that becomes $P_{ij}^{(\Omega)}(z, \varphi)$ when $z_1 = z_2 = \dots = z_i = z$. In paper (3) the following correction must be introduced: $\|P_{ij}^{(\Omega)}\|$ should everywhere be replaced by $\|P_{ij}^{(\Omega)*}\|$.

Introduce the notation: C is the Banach space of continuous bounded functions $\varphi(t)$ with values in Z , defined on $[0, \infty)$ ($\|\varphi\|_C = \sup_{t \in [0, \infty)} \|\varphi(t)\|_Z$); C_ν is the Banach space of continuous bounded functions $\varphi(t)$ with values in Z , defined on $[0, \infty)$, and such that there exists

$$\|\varphi\|_{C_\nu} = \sup_{t \in [0, \infty)} \|e^{\nu t} \varphi(t)\|_Z \quad (\nu > 0, \text{ see § 2});$$

$$\tilde{C} = [u; u = T(t)z_0; z_0 \in Z; \|u\|_{\tilde{C}} = \|u\|_C]; \quad (9)$$

obviously, $\tilde{C}$ is a subspace of the space C ;

$$\Lambda = \max_{x \geq 0} \frac{1}{N} \left[x - \frac{N}{\nu} \bar{f}(x) \right]; \quad (10)$$

$$S_1 = [z_0; z_0 \in Z; \|z_0\| < \Lambda]; \quad (11)$$

$$S_2 = [u; u \in \tilde{C}; \|u\|_C < N\Lambda]. \quad (12)$$

Lemma 1. Equation (8) has a solution $z(z_0, t)$ possessing the following properties:

- 1) $z(z_0, t)$ is a function analytic in three senses: (α) from S_1 into C ; (β) from S_1 into C_ν ; (γ) from S_2 into C .
- 2) $z(z_0, t)$, analytic in the senses (α) and (β), is majorized by the function $x_1(\lambda)$, and in the sense (γ) by the function $x_2(\lambda)$, where $x_1(\lambda)$ and $x_2(\lambda)$ are analytic solutions of the equations

$$x = N\lambda + \frac{N}{\nu} \bar{f}(x), \quad (13)$$

$$x = \lambda + \frac{N}{\nu} \bar{f}(x), \quad (14)$$

respectively, satisfying the condition $x_1(0) = x_2(0) = 0$.

- 3) If X is the value of x at which the maximum in (10) is attained, then

$$\|z(z_0, t)\|_{C_\nu} \leq X.$$

§ 4. As is known (2), if the operator A satisfies condition Ia) of § 2, then equation (8) and equation (1) with initial condition (2) are equivalent. Taking into account the uniqueness (2) of the solution of the differential problem (1)–(2), we obtain from Lemma 1 the following assertion:

Theorem 1. Let equation (1) satisfy conditions I of § 2. Then:

- 1) The trivial solution of equation (1) is asymptotically stable, and the domain of asymptotic stability contains the sphere $\|z_0\| < \Lambda$, where Λ is determined from (10).
- 2) If X is the value of x at which the maximum in (10) is attained, then, for solutions of equation (1) with initial conditions (2) from the sphere $\|z_0\| < \Lambda$, the inequality

$$\|z(z_0, t)\|_Z \leq X e^{-\nu t} \quad (t \geq 0)$$

holds.

- 3) The solution of the differential problem (1)–(2) is an analytic function of the vector of initial conditions z_0 from the sphere $\|z_0\| < \Lambda$ into the space C ; it is also an analytic function from the sphere $\|T(t)z_0\|_C < N\Lambda$ into the space C of the solution of the corresponding linear problem (4).
- 4) The analytic functions of the preceding item of the theorem are majorized, respectively, by the functions $x_1(\lambda)$ and $x_2(\lambda)$ (see Lemma 1).

§ 5. **Lemma 2.** Let conditions II of § 2 hold; $u_0 \in S_2$ (see (12)) and $du_0/dt \in C$ with $\|du_0/dt\|_C = m$.

Write the solution of equation (8) in the form

$$z(z_0, t) = \sum_{k=1}^{\infty} P_k(t)(u_0) \quad (15)$$

(see item 1 of Lemma 1).

Then the terms $P_k(t)(u_0)$ of the series (15) have derivatives

$$\frac{d}{dt} P_k(t)(u_0) \in C, \quad \left\| \frac{d}{dt} P_k(t)(u_0) \right\|_C \leq q_k \|u_0\|^k,$$

where q_k are the coefficients of the series

$$\sum_{k=1}^{\infty} q_k \lambda^k = \frac{m_0 \lambda}{1 - N v^{-1} f'_x [x_2(\lambda)]}. \quad (16)$$

Obviously, $q_k \geq 0$, and the series (16) converges for $|\lambda| < N\Lambda$. Here $m_0 = m/\|u_0\|$, and $x_2(\lambda)$ is as in Lemma 1.

Corollary. Suppose that the conditions II of § 2 are satisfied and that $z_0 \in D(A) \cap S_1$. Then:

- 1) The solution $z(z_0, t)$ of equation (8) has a continuous bounded derivative

$$\frac{d}{dt} z(z_0, t) \in C^*.$$

- 2) It is the unique solution of the differential problem (1)–(2). (Here the results of the works (7, 8) are used.)

Following the ideas of (9), we shall call a generalized solution of equation (1) with initial condition (2) the limit, as $n \rightarrow \infty$, in the space C , of a convergent sequence of solutions $\{z_n(t)\}$ of equation (1) satisfying the initial conditions z_{0n} , if, as $n \rightarrow \infty$, the sequence $\{z_{0n}\}$ converges to z_0 .

It is easy to see that, for $z_0 \in S_1$, the solution of equation (8) is the unique generalized solution of the differential problem (1)–(2).

Theorem 2. Suppose that the conditions II of § 2 are satisfied. Then all the assertions of Theorem 1 are valid for the generalized solution of equation (1) with initial condition (2), except that in item 3 of Theorem 1, instead of “...the solution of the corresponding linear problem...” one should read “...the generalized solution of the corresponding linear problem...” .

§ 6. In conclusion we give a simple example.

Consider (9, 10) the equation of one-dimensional diffusion (the diffusion coefficient is assumed constant), proceeding simultaneously with an irreversible third-order chemical reaction in an absorbing medium:

$$\partial u / \partial t = \partial^2 u / \partial x^2 + k u^3; \quad u(t, 0) = u(t, \pi) = 0; \quad u(0, x) = u_0(x). \quad (17)$$

Let Y be the set of real odd functions, defined on $[-\pi, \pi]$, which vanish at the endpoints of this interval and are such that the series of their Fourier coefficients converge absolutely. This set becomes a B -space if one introduces in it the norm

$$\|y\| = \sum_{k=1}^{\infty} |b_k^{(y)}|$$

$(b_k^{(y)})$ are the coefficients of the Fourier series of the element y).

If one assumes that $u_0(x) \in Y$ and takes for Z the complex extension of Y , then it is easy to show that in problem (17) the conditions II of § 2 are satisfied with $v = 1$, $N = 1$. Therefore all the assertions of Theorem 2 are valid with

$$\Lambda = \frac{2}{3\sqrt{3}|k|}; \quad X = \frac{1}{\sqrt{3}|k|}; \quad \|z(z_0, t)\| < \frac{1}{\sqrt{3}|k|} e^{-t}.$$

Leningrad Electrotechnical Institute
named after V. I. Ulyanov (Lenin)

Received
13 IV 1967

REFERENCES

1. A. M. Lyapunov, *The General Problem of the Stability of Motion*, Moscow–Leningrad, 1950.
2. M. G. Krein, *Lectures on the Theory of Stability of Solutions of Differential Equations in Banach Space*, Kiev, 1964.
3. A. E. Gelman, DAN, 132, No. 3 (1960).
4. E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, Moscow, 1962.
5. V. T. Maz' ya, P. E. Sobolevskii, UMN, 17, issue 6, 151 (1962).
6. P. E. Sobolevskii, Tr. Moscow Math. Soc., 10, 297 (1961).
7. Ya. D. Mamedov, P. E. Sobolevskii, Uchen. zap. Azerbaijan State Univ. named after S. M. Kirov, Ser. Phys.-Math., No. 3 (1963).
8. M. A. Krasnosel' skii, S. G. Krein, P. E. Sobolevskii, DAN, 111, No. 1 (1956).
9. I. G. Petrovskii, *Lectures on Partial Differential Equations*, Moscow, 1953.
10. J. Crank, *The Mathematics of Diffusion*, Oxford, 1956.
11. A. A. Zhukhovitskii, L. A. Shvartsman, *Physical Chemistry*, Moscow, 1964.

* If $z_0 \in D(A^n) \cap S_1$, then the solution $z(z_0, t)$ of equation (8) has $d^n z(z_0, t)/dt^n \in C$.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.