

ON FRACTIONAL INTEGRATION BY PARTS

P. E. Sobolevskii

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.61509>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.5

MATHEMATICS

P. E. SOBOLEVSKII

ON FRACTIONAL INTEGRATION BY PARTS

(Presented by Academician I. N. Vekua on March 1, 1968)

Consider the problem

$$v'(t) + Av(t) = f(t) \quad (0 \leq t \leq 2\pi), \quad v(0) = v_0 \quad (1)$$

in a Hilbert space H with a positive definite self-adjoint operator A . By its solution we shall mean a function $v(t)$ satisfying (1) such that the functions $v'(t)$ and $Av(t)$ are continuous. If a solution of problem (1) exists, then it is given by the formula

$$v(t) = \exp\{-tA\}v_0 + \int_0^t \exp\{-(t-s)A\}f(s) ds = \exp\{-tA\}v_0 + Qf(t), \quad (2)$$

where $\exp\{-tA\}$ is the semigroup generated by the operator A . Necessary conditions for the existence of a solution of problem (1) are the continuity of the function $f(t)$ and the membership of v_0 in the domain $D(A)$ of the operator A . Are these conditions sufficient for the existence of a solution of problem (1)? If $v_0 \in D(A)$, then problem (1) has a solution in the case when the problem

$$v'(t) + Av(t) = f(t) \quad (0 \leq t \leq 2\pi), \quad v(0) = 0. \quad (3)$$

has a solution.

In the present paper, the solvability of problem (3) is proved under the assumption that $f(t)$ belongs to the abstract space $W_2^{1/2}$ of L. N. Slobodetskii. We note that $f(t) \in W_2^{1/2}$ need not satisfy any Hölder condition. This result is based on the method of estimating integrals of the form

$$I_i = \int_{\Omega} u \frac{\partial v}{\partial x_i} dx, \quad i = 1, \dots, n, \quad (4)$$

of products of smooth scalar functions $u(x)$ and $v(x)$ (or functions with values in a Hilbert space) defined in an n -dimensional domain Ω .

1. Theorem 1. Let the boundary S of the domain Ω be continuously differentiable once. Then the inequality

$$|I_i| \leq M(\alpha) \|u\|_{W_2^\alpha(\Omega)} \|v\|_{W_2^{1-\alpha}(\Omega)} \quad (5)$$

holds for any α from $[0, 1/2)$, and the inequality

$$|I_i| \leq M(1/2) (\|u\|_{W_2^{1/2}(\Omega)} + \|u\|_{L_2(S)}) (\|v\|_{W_2^{1/2}(\Omega)} + \|v\|_{L_2(S)}). \quad (6)$$

We shall prove, for the one-dimensional integral

$$I = \int_0^{2\pi} u(x) v'(x) dx,$$

the one-dimensional analogues of these inequalities,

$$|I| \leq M(\alpha) \|u\|_{W_2^\alpha} \|v\|_{W_2^{1-\alpha}} \quad (0 \leq \alpha < 1/2), \quad (7)$$

$$|I| \leq M(1/2) (\|u\|_{W_2^{1/2}} + |u(0)| + |u(2\pi)|) (\|v\|_{W_2^{1/2}} + |v(0)| + |v(2\pi)|). \quad (8)$$

Consider inequality (7). Let the function $u(x)$ be periodic ($u(0) = u(2\pi)$). Setting $v(x) = v_1(x) + v_2(x)$, where $v_2(x) = (x/2\pi)[v(2\pi) - v(0)]$, we obtain

$$I = I_1 + I_2 = \int_0^{2\pi} u(x) v_1'(x) dx + \frac{1}{2\pi} \int_0^{2\pi} u(x) dx [v(2\pi) - v(0)].$$

Let

$$u(x) = \sum_{|n|=0}^{+\infty} a_n e^{inx}, \quad v_1(x) = \sum_{|n|=0}^{+\infty} b_n e^{inx}.$$

Then

$$|I_1| \leq 2\pi \sum |n| |a_n| |b_n| \leq 2\pi \left(\sum |n|^{2\alpha} |a_n|^2 \right)^{1/2} \left(\sum |n|^{2-2\alpha} |b_n|^2 \right)^{1/2}$$

But

$$2\pi \sum (1 + |n|^{2\alpha}) |a_n|^2 = \|u\|_{W_2^\alpha}^2, \quad 2\pi \sum (1 + |n|^{2-2\alpha}) |b_n|^2 = \|v_1\|_{W_2^{1-\alpha}}^2,$$

since the functions $u(x)$ and $v_1(x)$ are periodic. Therefore

$$|I_1| \leq \|u\|_{W_2^\alpha} \|v_1\|_{W_2^{1-\alpha}}.$$

Further, for $\alpha > 0$,

$$\|v_1\|_{W_2^{1-\alpha}} \leq \|v\|_{W_2^{1-\alpha}} + \|v_2\|_{W_2^{1-\alpha}} \leq \|v\|_{W_2^{1-\alpha}} + C_1(\alpha) |v(2\pi) - v(0)|.$$

By the embedding theorems,

$$|v(x)| \leq C_2(\alpha) \|v\|_{W_2^{1-\alpha}}$$

for $\alpha < 1/2$. Finally,

$$\left| (1/2\pi) \int_0^{2\pi} u(x) dx \right| \leq C_3 \|u\|_{W_2^\alpha}.$$

Hence inequality (7) follows for a periodic smooth function $u(x)$. However, for $\alpha < 1/2$ a smooth function $u(x)$ can be approximated in the metric W_2^α by smooth periodic functions. Obviously, it is enough to prove this assertion for the linear function $(t/\pi)[u(2\pi) - u(0)]$, which can be approximated in the metric W_2^α , for $\alpha < 1/2$, by its partial Fourier sums (these partial sums are periodic functions, and their Fourier coefficients decrease as c/n).

To prove inequality (8), the functions $u(x)$ and $v(x)$ must be reduced to periodic ones by the method indicated above.

2. The closure of the set K of all smooth functions $f(t)$, defined on $[0, 2\pi]$ with values in H , in the norm

$$\|f\|_{L_2} = \left(\int_0^{2\pi} \|f\|_H^2(t) dt \right)^{1/2}$$

forms the space $L_2 = L_2([0, 2\pi], H)$. The closure of the set K in the norm

$$\|f\|_{W_2^1} := (\|f\|_{L_2}^2 + \|f'\|_{L_2}^2)^{1/2}$$

forms the space W_2^1 . The closure of the set K in the norm

$$\|f\|_{W_2^\alpha} = \left(\|f\|_{L_2}^2 + \int_0^{2\pi} \int_0^{2\pi} \|f(t) - f(s)\|_H^2 |t - s|^{-1-2\alpha} dt ds \right)^{1/2} \quad (0 < \alpha < 1)$$

forms the space W_2^α . The spaces W_2^l are defined analogously for any (integer or fractional) l .

If in the preceding constructions one considers, instead of K , its subset $\dot{K}$, consisting of functions $f(t)$ for which $f(0) = 0$, then we obtain the spaces $\dot{L}_2, \dot{W}_2^1, W_2^\alpha$. Obviously, $\dot{L}_2 = L_2$. Just as in item 1, it is shown that $\dot{W}_2^\alpha = W_2^\alpha$ for $\alpha < 1/2$.

It is known (see, for example, (1)) that the operators AQ and $\frac{d}{dt}Q$ act boundedly in the spaces L_2 and $\dot{W}_2^1$. Hence it follows that

Theorem 2. The operators AQ and $\frac{d}{dt}Q$ act boundedly in the spaces $\dot{W}_2^\alpha$.

For the proof one may use the method applied in (2), p. 158, item 7, for spaces of scalar functions. In doing so, instead of the space $L_2((-\infty, +\infty), H)$ one must take its subspace L_2 , consisting of odd functions, and take into account that the Fourier transform is a unitary operator in L_2 , that in it there acts the operator J of multiplication by the function $(1 + |t|^{\frac{1}{2}})^{1/2}$, and that it is a positive definite self-adjoint operator. Next, the domains of definition of the fractional powers J^α of the operator J form the spaces $\dot{W}_2^\alpha$. Finally, the traces of functions from these spaces on the segment $[0, 2\pi]$ form the spaces $\dot{W}_2^\alpha$ introduced above. Therefore between the spaces L_2 and $\dot{W}_2^1$ one can stretch a Hilbert scale H_α so that the norms in H_α and in $\dot{W}_2^\alpha$ are equivalent. Applying the interpolation theorem, we obtain the assertion of Theorem 2.

From Theorem 2 and the interpolation theorem it follows that

Theorem 3. The operator $A^\alpha Q$ acts boundedly from L_2 to $W_2^{1-\alpha}$ and from $\dot{W}_2^\beta$ to $\dot{W}_2^{\beta+1-\alpha}$.

3. Denote by $C^\alpha = C^\alpha([0, 2\pi], H)$ ($0 \leq \alpha \leq 1$) the closure of the set K in the norm

$$\|f\|_{C^\alpha} = \max_{0 \leq t \leq 2\pi} \|f(t)\|_H + \sup_{0 \leq t, s \leq 2\pi} |t - s|^{-\alpha} \|f(t) - f(s)\|_H.$$

Theorem 4. For any α from $[0, 1]$, $T > 0$, v_0 from H , the function $v(t) = \exp\{-tA\}A^{-\alpha}v_0$ belongs to $C^\alpha([0, T], H) \cap W_2^\beta([0, T], H)$ for $\beta = (1 + \alpha)/2$, and the inequality

$$\|v(t)\|_{C^\alpha \cap W_2^\beta} \leq M_\alpha \|v_0\|_H, \quad (9)$$

holds, where M_α depends neither on T nor on v_0 . Here $\|\cdot\|_{C^\alpha \cap W_2^\beta} = \|\cdot\|_{C^\alpha} + \|\cdot\|_{W_2^\beta}$.

Proof. If $v_0 \in D(A^n)$, then the function $v(t)$ is n times continuously differentiable. Since $D(A^n)$ is dense in H , for the proof of the theorem it suffices to establish estimate (9). To prove the estimate $\|v(t)\|_{C^\alpha} \leq M_\alpha \|v_0\|_H$ one must use the spectral decomposition

$$A = \int_\delta^{+\infty} \lambda dE_\lambda, \quad \delta > 0$$

and the representation, following from it, for the function $\|v(t) - v(\tau)\|_H^2$

$$\|v(t) - v(\tau)\|_H^2 = \int_\delta^{+\infty} [e^{-t\lambda} - e^{-\tau\lambda}]^2 \lambda^{-2\alpha} d(E_\lambda v_0, v_0)_H.$$

Estimation of the integrand leads to the required inequality. Consider the integral

$$\begin{aligned} I &= \int_0^T \int_0^T \|v(t) - v(\tau)\|_H^2 |t - \tau|^{-1-2\beta} dt d\tau = \\ &= \int_\delta^{+\infty} \left[\int_0^T \int_0^T |e^{-t\lambda} - e^{-\tau\lambda}|^2 \lambda^{-2\alpha} |t - \tau|^{-1-2\beta} dt d\tau \right] d(E_\lambda v_0, v_0). \end{aligned}$$

Further,

$$I = \int_\delta^{+\infty} \left[\int_0^{T\lambda} \int_0^{T\lambda} |e^{-s} - e^{-z}|^2 |s - z|^{-1-2\beta} ds dz \right] d(E_\lambda v_0, v_0) \leq M_\alpha^2 \|v_0\|_H^2,$$

since the inner integrals are bounded uniformly in λ .

Corollary 1. The operators AQ and $\frac{d}{dt}Q$ act boundedly from

$$W_2^{1/2} \cap C^0 \quad \text{to} \quad W_2^{1/2}.$$

Theorem 5. The operator $A^\beta Q$ acts boundedly from W_2^α to $C^{1/2+\alpha-\beta}$ for $0 \leq \alpha < 1/2$ and $0 \leq \beta \leq 1/2 + \alpha$. The operator $A^\beta Q$ acts boundedly from $W_2^{1/2} \cap C^0$ to $C^{1-\beta}$ for $0 \leq \beta \leq 1$.

Proof. If $\alpha < \frac{1}{2}$ and $\beta < \frac{1}{2} + \alpha$, then the result follows from the embedding of the space W_2^α into the space $L_{2/(1-2\alpha)}$ and from the estimate

$$\|A^\alpha[\exp\{-(t+\Delta t)A\} - \exp\{-tA\}]\|_{H \rightarrow H} \leq C(\alpha, \beta) \Delta t^\beta t^{-\alpha-\beta} e^{-\delta t} \\ (0 < t < t + \Delta t, \quad 0 \leq \beta \leq 1, \quad \alpha > -\beta), \quad (10)$$

which follows from the integral representation standing under the sign of the $\|\cdot\|_{H \rightarrow H}$ operator norm.

If, however, $\beta = \frac{1}{2} + \alpha$ and $\alpha < \frac{1}{2}$, then to study the operator $A^\beta Q$ we apply the methods of Sec. 1. Let v_0 be an arbitrary element of H . Put

$$\varphi(t) = (A^\beta Q f(t), v_0)_H = \int_0^t (A^\beta \exp\{-(t-s)A\} f(s), v_0)_H ds \\ = \int_0^t (f(s), A \exp\{-(t-s)A\} A^{\beta-1} v_0)_H ds.$$

Here the self-adjointness of the operator A has been used. Further,

$$\varphi(t) = \int_0^t \left(f(s), \frac{d}{ds} [\exp\{-(t-s)A\} A^{\beta-1} v_0] \right)_H ds = \int_0^t (f(s), \psi'(s))_H ds.$$

Let $\psi_1(s) = \psi(s) - \psi_2(s)$, where $\psi_2(s) = (s/t)[\psi(t) - \psi(0)]$. Then $\psi_1(s)$ is already periodic on $[0, t]$. Let also $f(s)$ be periodic on $[0, t]$. Then for the function

$$\varphi_1(t) = \int_0^t (f(s), \psi_1'(s))_H ds$$

the estimate $|\varphi_1(t)| \leq \|f\|_{W_2^\alpha} \|\psi_1\|_{W_2^{1-\alpha}}$ is valid at the point t . Here the spaces W_2 are considered on the segment $[0, t]$. Further, $\|\psi\|_{W_2^{1-\alpha}} \leq C_1 \|v_0\|_H$. Here Theorem 4 is applied. With the aid of (10) we obtain $\|\psi_2\|_{W_2^{1-\alpha}} \leq C_2 \|v_0\|_H$. Therefore $\|\psi_1\|_{W_2^{1-\alpha}} \leq C_3 \|v_0\|_H$. Hence it follows that $|\varphi_1(t)| \leq C_4 \|f\|_{W_2^\alpha} \|v_0\|_H$. By virtue of (10),

$$|\varphi(t) - \varphi_1(t)| \leq C_5 t^{-1/2-\alpha} \int_0^t \|f(s)\|_H ds \leq C_6 \|f\|_{L_{2/(1-2\alpha)}} \leq C_7(\alpha) \|f\|_{W_2^\alpha}.$$

In the last inequality the embedding of W_2^α in $L_{2/(1-2\alpha)}$ has been used. Thus, finally, we obtain that $|\varphi(t)| \leq M(\alpha) \|f\|_{W_2^\alpha} \|v_0\|_H$. By means of passage to the

limit we free ourselves (since $\alpha < \frac{1}{2}$) from the assumption of periodicity of $f(s)$ on $[0, t]$.

Let now $\alpha = \frac{1}{2}$ and $\beta = 1$. Then the estimate of the function $AQf(t)$ is established as above in the case $\beta = \frac{1}{2} + \alpha$ with $\alpha < \frac{1}{2}$, but first one has to pass to a periodic function. From the estimate of the function $AQf(t)$ follows the estimate of the function $\frac{d}{dt}Qf(t)$, since the function $Qf(t)$ satisfies equation (3). Hence, from the multiplicative inequality for Hölder norms, the second assertion of Theorem 5 follows.

Corollary 2. The operators AQ and $\frac{d}{dt}Q$ act boundedly in the space $W_2^{1/2} \cap C^0$.

Theorem 6. In order that problem (1) have such a solution $v(t)$ that the functions $v'(t)$ and $Av(t)$ belong to the space $W_2^{1/2} \cap C^0$, it is necessary and sufficient that v_0 belong to $D(A)$ and $f(t)$ belong to $W_2^{1/2} \cap C^0$.

Voronezh State University

Received
23 II 1968

REFERENCES

1. P. E. Sobolevskii, DAN, 122, No. 6 (1958).
2. S. G. Krein, Yu. I. Petunin, UMN, 21, issue 2 (128), 89 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.