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Abstract

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MATHEMATICS

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ON THE BUBNOV-GALERKIN METHOD FOR PARABOLIC EQUATIONS IN A HILBERT SPACE

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1. In a Hilbert space H we consider the problem

$$v' + A(t)v = f(t, v) \quad (0 \leq t \leq T), \quad v(0) = v_0. \quad (1)$$

Its generalized solutions were studied in ⁽¹⁾. An **approximate solution** of problem (1) is the generalized solution of the problem

$$v'_n + P_n A(t)v_n = P_n f(t, v_n) \quad (0 \leq t \leq T), \quad v_n(0) = P_n v_0. \quad (2)$$

Here P_n is the operator of orthogonal projection onto the linear span H_n of the first n elements of the system $\{e_i\}$.

It is assumed that $A(t)$, for each $t \in [0, T]$, is a positive definite self-adjoint operator whose domain D does not depend on t . It is assumed that the following hold: (a) $A(t)A^{-1}(0)$ has only discontinuities of the first kind; (b) $A(t)A^{-1}(0)$ has bounded variation; (c) $A(t)A^{-1}(0)$ satisfies a Hölder condition with exponent $\varepsilon > 0$.

It is assumed that all $e_i \in D$ and (in the convergence theorems) that the system $\{e_i\}$ is complete in the energy norm, i.e., that the set

$$\bigcup_{n=1}^{\infty} A^{1/2}(0)H_n$$

is dense in H .

Below we give results on problem (2), obtained in 1957-1958 ⁽²⁾.

2. Since $P_n A(t)$ is a positive definite operator in H_n and $H_n \subset D$, for any $0 \leq \alpha \leq 1$, $0 \leq t \leq T$, $n = 1, 2, \dots$, the functions

$$\varphi_\alpha = \varphi_\alpha(n, t) = \|A^\alpha(t)[P_n A(t)]^{-\alpha} P_n\|,$$

$$\psi_\alpha = \psi_\alpha(n, t) = \|[P_n A(t)]^\alpha P_n A^{-\alpha}(t)\|$$

are defined.

Lemma 1. The functions φ_α and ψ_α are logarithmically convex for $0 \leq \alpha \leq 1$; $\varphi_\alpha \leq 1$ for $0 \leq \alpha \leq 1/2$. If $\varphi_1 \leq C_1$, then $\varphi_\alpha \leq C_1^{2\alpha-1}$ for $1/2 \leq \alpha \leq 1$. If $\psi_{\alpha_0} \leq D_{\alpha_0}$ for some α_0 from $[0, 1]$, then $\psi_\alpha \leq D_{\alpha_0}^{\alpha/\alpha_0}$ for all α from $[0, \alpha_0]$.

The proof is based on Heinz inequalities (see ^(3,4) and ⁽⁵⁾, p. 205).

The estimate $\varphi_{1/2} = 1$ was used in ⁽⁶⁾. The estimate $\varphi_1 \leq C_1$ holds if there exists an invertible operator B such that $D(B) = D$,

$$C_1(A(t)v, Bv) \geq \|A(t)v\| \cdot \|Bv\| \quad \text{for } v \in D,$$

and H_n are invariant with respect to B ⁽⁷⁾. If B^{-1} is bounded, then the estimate $\varphi_1 \leq C_1$ means that for the systems $\{e_i\}$ and $\{AB^{-1}e_i\}$ the condition (A) of N. I. Pol'skii is satisfied (see ^(10,8), p. 124). The estimate $\psi_{\alpha_0} \leq D_{\alpha_0}$ in the case $\alpha_0 = 1/2, 1$ holds if there exists such a bounded invertible operator B_{α_0} that

$$D(B_{\alpha_0}) = D(A^{\alpha_0}(0)) = D(A^{\alpha_0}(t)),$$

and H_n are invariant with respect to B_{α_0} .

3. Denote by $U_n(t, s)$ ($0 \leq s \leq t \leq T$) the operator such that the function $v_n = U_n(t, s)v_{0n}$ is the solution of the homogeneous problem

$$v'_n + P_n A(t)v_n = 0 \quad (s \leq t \leq T), \quad v_n(s) = v_{0n} \in H_n. \quad (3)$$

With the aid of the methods developed in [1] and the estimates of item 2, the following is established:

Lemma 2. Suppose that (a) is satisfied and $\varphi_1 \leq C$. Then for every $v \in H_n$ the estimates

$$\begin{aligned} \max \left\{ \|[P_{nA}(0)]^{1/2} U_n(t, s)v\|, \left(\int_0^t \|P_{nA}(\tau) U_n(\tau, s)v\|^2 d\tau \right)^{1/2} \right\} \leq \\ \leq K \|[P_{nA}(0)]^{1/2} v\| \end{aligned} \quad (4)$$

are valid.

Lemma 3. The estimates

$$\| [P_{nA}(t)]^\beta U_n(t, s) [P_{nA}(s)]^{-\alpha} P_n \| \leq K |t - s|^{\alpha - \beta} \quad (5)$$

hold for $0 \leq \alpha \leq \beta \leq 1/2$, if (b) is satisfied; for $0 \leq \alpha \leq \beta < 1/2 + \varepsilon$, if (c) is satisfied; for $0 \leq \alpha \leq 1$, $0 \leq \beta < 1 + \varepsilon$, $\alpha \leq \beta$, if (c) is satisfied and $\varphi_1 \leq C_1$.

4. The uniform (in n and t) estimates of items 2 and 3 make it possible to establish the convergence of the solutions of problem (3) (for $v_{0n} = P_n v_0$) to the solution $v(t) = U(t, s)v_0$ of the problem

$$v' + A(t)v = 0 \quad (s \leq t \leq T), \quad v(s) = v_0. \quad (6)$$

First of all, the limiting relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} \|U_n(t, s)v_m - U(t, s)v_m\| = 0 \quad (7)$$

is established. For this purpose it is first shown that $U_n(t, s)x_m$ converges weakly to $U(t, s)x_m$. The proof of this fact is based on the inequalities

$$\|A^{1/2}(0)U_n(t, s)v_m\| \leq K \|A^{1/2}(0)v_m\|,$$

$$\| [U_n(t + \Delta t, s + \Delta s) - U(t, s)]v_m \| \leq [|\Delta t|^{1/2} + |\Delta s|^{1/2}] K \|A^{1/2}(0)v_m\|, \quad (8)$$

which follow from Lemmas 1-3. Then the convergence of $\|U_n(t, s)x_m\|$ to $\|U(t, s)x_m\|$ is proved. For this one uses the weak convergence of $\widetilde{U}_n(2t - s, s)v_m$ to $\widetilde{U}(2t - s, s)v_m$, constructed from the operator

$$\widetilde{A}(\tau) = \begin{cases} A(\tau), & s \leq \tau \leq t, \\ A(2t - \tau), & t \leq \tau \leq 2t, \end{cases} \quad (9)$$

and the formulas $\widetilde{U}_n(2t - s, s) = U_n^*(t, s)U_n(t, s)$, $\widetilde{U}(2t - s, s) = U^*(t, s)U(t, s)$. Further, by means of (7), Lemmas 1-3, and the moment inequality (9), the following are established:

Theorem 1. For every $v_0 \in H$ the limiting relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^\alpha \|A^\alpha(0)[U_n(t, s)P_n v_0 - U(t, s)v_0]\| = 0 \quad (10)$$

holds for $\alpha = 0$, if (a) is satisfied and $\varphi_1 \leq C_1$; for $0 \leq \alpha < 1/2$, if (b) is satisfied; for $0 \leq \alpha \leq 1/2$, if (c) is satisfied; for $0 \leq \alpha < 1$, if (b) or (c) is satisfied and $\varphi_1 \leq C_1$.

In applications to partial differential equations, (10) means convergence of the solutions together with their derivatives with respect to the spatial variables.

Theorem 2. For every $v_0 \in H$ the limiting relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s| \left\| \frac{\partial}{\partial t} U_n(t, s) P_n v_0 - \frac{\partial}{\partial t} U(t, s) v_0 \right\| = 0, \quad (11)$$

holds, if (c) is satisfied and $\varphi_1 \leq C_1$.

If $v_0 \in D(A^\gamma(0))$, then the preceding assertions can be sharpened.

Theorem 3. For every $v_0 \in D(A^{1/2}(0))$ the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} \|A^\alpha(0) [U_n(t, s) P_n v_0 - U(t, s) v_0]\| = 0, \quad (12)$$

holds, if (a) is satisfied, $\varphi_1 \leq C_1$ and $\psi_{1/2} \leq D_{1/2}$.

Theorem 4. For every $v_0 \in D(A^\gamma(0))$ the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^{\alpha - \gamma} \|A^\alpha(0) [U_n(t, s) P_n v_0 - U(t, s) v_0]\| = 0 \quad (13)$$

holds for $0 \leq \gamma \leq \alpha < 1$, if (b) or (c) is satisfied, $\varphi_1 \leq C_1$ and $\psi_\gamma \leq D_\gamma$.

Theorem 5. For every $v_0 \in D(A^\gamma(0))$ the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^{1 - \gamma} \left\| \frac{\partial}{\partial t} U_n(t, s) P_n v_0 - \frac{\partial}{\partial t} U(t, s) v_0 \right\| = 0 \quad (14)$$

holds for $0 \leq \gamma \leq 1$, if (c) is satisfied, $\varphi_1 \leq C_1$ and $\psi_\gamma \leq D_\gamma$.

Finally, we formulate conditions under which not only the solutions v_n of problems (3) (with $v_{0n} = P_n v_0$) converge to the solution v of problem (6), and not only the derivatives $v'_n \rightarrow v'$ (Theorems 2 and 5), but also the residuals in equation (6) tend to zero.

Theorem 6. For every $v_0 \in D(A^\gamma(0))$ the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} |t - s|^{1 - \gamma} \|A(t) [U_n(t, s) P_n v_0 - U(t, s) v_0]\| = 0 \quad (15)$$

holds for $0 \leq \gamma \leq 1$, if (c) is satisfied, $\varphi_1 \leq C_1$, and there exists an operator B having an inverse such that $D(B) = D$ and the H_n are invariant with respect to B .

Recall (Section 2) that $\varphi_1 \leq C_1$, if $C_1(A(t)v, Bv) \geq \|A(t)v\| \cdot \|Bv\|$ for $v \in D$.

5. Since (see (1)) the solution of problem (2) is a solution of the equation

$$v_n(t) = U_n(t, 0)P_n v_0 + \int_0^t U_n(t, s)P_n f[s, v_n(s)] ds, \quad (16)$$

the assertions given in Section 4 make it possible to study the convergence of the solutions of problem (6) to the solution of problem (1).

We first consider the case of a linear equation. We shall state only the theorem on the convergence of the residuals to zero.

Theorem 7. Suppose that the conditions of Theorem 6 are satisfied. Then the limit relation

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \left\| (I - P_n)A \int_0^t U_n(t, s)P_n f(s) ds \right\| = 0 \quad (17)$$

holds for any function $f(t)$ satisfying the Hölder condition.

We now proceed to the consideration of the nonlinear equation (16).

Theorem 8. There exists a segment $[0, h] \subset [0, T]$, independent of n , such that equation (16) has at least one solution $v_n(t)$, defined on $[0, h]$. The family $A^\alpha(0)v_n(t)$ is compact in $C([0, T], H)$. Its limit points have the form $A^\alpha(0)v(t)$, where $v(t)$ are generalized solutions of problem (1). If problem (1) has a unique solution $v(t)$, then

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq h} \|A^\alpha(0)[v_n(t) - v(t)]\| = 0. \quad (18)$$

These assertions hold for some $\alpha \in [0, 1/2)$, if (a) is satisfied, the operator $A^{-1}(0)$ is completely continuous, the operator $f(t, A^{-\alpha}(0)w(t))$ acts boundedly and continuously from $C([0, T], H)$ into $B_2([0, T], H)$, $v_0 \in D(A^{1/2}(0))$, $\varphi_1 \leq C_1$, and $\psi_{1/2} \leq D_{1/2}$; and for some $\alpha \in [0, 1)$, if (b) or (c) is satisfied, the operator $A^{-1}(0)$ is completely continuous, the operator $f(t, A^{-\alpha}(0)w(t))$ acts boundedly and continuously from $C([0, T], H)$ into $B_p([0, T], H)$ for some $p > 1/(1 - \alpha)$, $v_0 \in D(A^\gamma(0))$ for some $\alpha < \gamma \leq 1$, $\varphi_1 \leq C_1$, and, if $\alpha \geq 1/2$, $\psi_\gamma \leq C_\gamma$.

Finally, if the hypotheses of Theorem 6 are satisfied and the nonlinear operator $f(t, v)$ is sufficiently smooth, then the residuals converge to zero (uniformly in t).

6. In the preceding sections, the convergence, uniform in t , of the solutions of problem (2) to the solution of problem (1) was investigated. The use of the estimates from (1) makes it possible to study convergence in the space $B_2([0, T], H)$, and in some cases also in an arbitrary space $B_p([0, T], H)$.

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