

INFINITE HERMITIAN MATRICES WITH OPERATOR COEFFICIENTS

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Abstract

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MATHEMATICS

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INFINITE HERMITIAN MATRICES WITH OPERATOR COEFFICIENTS

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This article is devoted to the study of the deficiency indices of symmetric operators generated in a certain Hilbert space $\mathfrak{H}$ by infinite Hermitian matrices with operator coefficients.

1. Let $\{H_k\}_{k=0}^{\infty}$ be a sequence of Hilbert spaces with scalar products $(\cdot, \cdot)_k$ and norms $\|\cdot\|_k$. The Hilbert space $\mathfrak{H}$ is defined as the direct sum

$$\mathfrak{H} = \sum_{k=0}^{\infty} \oplus H_k.$$

The elements of this space are written in the form of sequences $U = \{u_0, u_1, \dots, u_k, \dots\}$, where $u_k \in H_k$. The scalar product in $\mathfrak{H}$ is defined by the formula

$$[U, V] = \sum_{k=0}^{\infty} (u_k, v_k)_k.$$

Linear operators in $\mathfrak{H}$ are specified by infinite matrices

$$C = (C_{jk})_{j,k=0}^{\infty}$$

with operator coefficients, where C_{jk} is a linear operator with domain of definition $D(C_{jk})$, dense in the space H_k , and range in H_j . Moreover, for all j and k , $D(C_{jk}) \supset D(C_{kk})$. The matrices are assumed to be **Hermitian** and **finite**. The Hermitian property of C means that $C_{kj} = C_{jk}^*$ (where C_{jk}^* is the operator adjoint to C_{jk}). Finiteness means that for each j there exists an integer $m_j \geq 0$ such that $C_{jk} = 0$ for all k satisfying $|j - k| > m_j$. Matrices for which $m_j \equiv m < \infty$ and for all j there exist bounded operators $C_{j+m,j}^{-1}$ will be called operator C_m -matrices.

Defined on the set of finite vectors* with components $u_k \in D(C_{kk})$, the correspondence $U \rightarrow CU$ generates a symmetric operator in $\mathfrak{H}$. Its closure in $\mathfrak{H}$ will be denoted by C . The operator C , obviously, is symmetric with a domain of definition dense in $\mathfrak{H}$.

Below we study the deficiency indices of the operator C depending on the properties of the operators C_{jk} . The formulated theorems find application in the consideration of operators in the space of secondary quantization.

2. Known sufficient conditions for self-adjointness of the operator C (⁽¹⁾, p. 151; ⁽²⁾) are a generalization of T. Carleman's condition, formulated for ordinary Jacobi matrices (C_1 -matrices with numerical coefficients). The theorem given below is a supplement to the results of ^(1,2) in the case when the operators $C_{k,k+j}$ ($j \neq 0$) are unbounded for every $j \neq 0$, or are bounded and for at least one $j \neq 0$ satisfy the condition

$$\sum_{k=0}^{\infty} \|C_{k,k+j}\|^{-1} < \infty.$$

* A vector is called **finite** if only a finite number of its components are different from zero.

Let I_k be the identity operator in H_k , and δ_{jk} the Kronecker symbol.

Theorem 1. *The operator C is self-adjoint if there exists a complex number σ such that the quantities*

$$\mu_{jk} = \|C_{jk}(C_{kk} - \sigma I_k)^{-1}\|$$

satisfy the estimates

$$\sup_k \sum_{j=0}^{\infty} (1 - \delta_{jk}) \mu_{jk} < 1, \quad \sup_j \sum_{k=0}^{\infty} (1 - \delta_{jk}) \mu_{jk} < 1$$

and, uniformly in k ,

$$\sum_{j=k+N}^{\infty} \mu_{jk} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

The proof of Theorem 1 is based on the well-known lemma of T. Kato (⁽⁴⁾). It asserts that, under a perturbation of a self-adjoint operator T_0 by a symmetric operator V , the property of self-adjointness is preserved if the domains of definition of the operators T_0 and V are related by the inclusion $D(T_0) \subset D(V)$, and there exist constants a and b ($0 \leq a < 1$, $b > 0$) such that for every $f \in D(T_0)$ the inequality

$$\|Vf\| \leq a\|T_0 f\| + b\|f\|$$

holds.

An illustration of Theorem 1 may be the case when $H_k = L_2(-\infty, \infty)$ for all $k \geq 0$ and the operator C is generated by an operator C_m -matrix: $C_{kk} = -d^2/dx^2$, while for $k < j \leq k + m$

$$C_{kj} + \varphi_{kj}(x) \frac{d}{dx} + \psi_{kj}(x)$$

and $C_{jk} = C_{kj}^*$. All the indicated operators are defined in $L_2(-\infty, \infty)$ on closures of finite functions, and the complex-valued functions $\varphi_{kj}(x)$, $\varphi'_{kj}(x)$, and $\psi_{kj}(x)$ belong to $L_2(-\infty, \infty)$, and their norms are uniformly bounded with respect to all k and j .

3. The presence of nonzero deficiency indices of the operator C is established for the case when all spaces H_k coincide and all operators C_{jk} are bounded. Put $\dim H_k = d \leq \infty$. The following theorem generalizes the corresponding results of papers (^{5, 6}).

Theorem 2. *Let the elements of the C_m -matrix be such that:*

- 1) $\sum_{k=0}^{\infty} \|C_{k,k+m}^{-1}\| < \infty$;
- 2) for sufficiently large k $\|C_{k,k+m}^{-1} C_{k,k-m}\| \leq 1 - \nu/k$, where $\nu > m$;
- 3) $\sum_{k=0}^{\infty} \|C_{k,k+m}^{-1} C_{k,k+j}\| < \infty$ for $|j| < m$.

Then the operator C has deficiency indices (dm, dm) .

4. Let now C belong to the class of C_1 -matrices with matrix coefficients ($m \times m$ -matrices), i.e. have the form

$$C_0 = \begin{pmatrix} A_0 & B_2 & & \\ B_0^* & A_1 & B_1 & \\ & & \ddots & \ddots \\ & & B_1^* & \ddots \end{pmatrix}, \quad (1)$$

where the unspecified elements are zero, and $\det B_k \neq 0$ ($k = 0, 1, \dots$). For such matrices, called by M. G. Krein (⁷) J_m -matrices, it is possible to supplement the results formulated in Theorems 1 and 2.

Let us introduce into consideration the equation with respect to $w(k)$

$$\det [B_{k-1}^* + (A_k - \lambda I)w(k) + B_k w^2(k)] = 0, \quad (2)$$

where λ is a complex number, and I is the identity matrix of order m . Number the roots of equation (2) in the order of increase of their moduli:

$$|w_1(k)| \leq |w_2(k)| \leq \dots \leq |w_{2m}(k)|.$$

Denote by n_+ (n_-) the number of solutions $w(k)$ of equation (2) for $\lambda = i$ ($\lambda = -i$) satisfying, for all sufficiently large k , the estimate $|w(k)| \leq 1 - \nu/k$, and by p_+ (p_-) –

the number of solutions with the estimate $|w(k)| \geq 1 - 1/2k$. In view of the condition $\det B_k \neq 0$, equation (2) is equivalent to the equation

$$\det [w^2(k)I + w(k)F_1(k; \lambda) + F_2(k)] = 0,$$

where $F_1(k; \lambda) = B_k^{-1}(A_k - \bar{\lambda}I)$, $F_2(k) = B_k^{-1}B_{k-1}^*$.

Theorem 3. Suppose: 1) the elements of the matrices $F_1(k+1; \pm i) - F_1(k; \pm i)$ and $F_2(k+1) - F_2(k)$ are absolutely summable on the interval $(0 \leq k \leq \infty)$; 2) the limiting values of all $2m$ roots $w_j = \lim_{k \rightarrow \infty} w_j(k)$ are nonzero and distinct; 3) there exists $\nu > 1/2$ such that $n_{\pm}^{\nu} + p_{\pm} = 2m$. Then the defect indices of the operator C are equal to $(m - p_+, m - p_-)$.

The proof of Theorem 3 is based on computing asymptotic estimates, as $k \rightarrow \infty$, for the components of the vector $u_k = (x_{km}, x_{km+1}, \dots, x_{km+m-1})$, which is a solution of the equation

$$B_{k-1}^* u_{k-1} + A_k u_k + B_k u_{k+1} = \bar{\lambda} u_k \quad (k \geq 1). \quad (3)$$

The plan of the proof is as follows: replace the system of second-order difference equations (3) by a system of first-order difference equations $y(k+1) = G_{ky}(k)$ (where $y(k)$ is a $2m$ -dimensional vector), and reduce the resulting system to quasidiagonal form

$$t(k+1) = (\Gamma_k + \Gamma_k D_k) t(k), \quad (4)$$

where Γ_k is a diagonal matrix, and the elements $d_{ij}(k)$ of the matrix D_k satisfy the condition

$$\sum_{k=0}^{\infty} |d_{ij}(k)| < \infty.$$

After this, it remains to use I. M. Rapoport's theorem ((⁸), p. 62) on the asymptotics of the solutions of equation (4) and to compute, for large k , the asymptotics of the products $\prod_{s=0}^k w_j(s)$, where $j = 1, 2, \dots, 2m$. This makes it possible to determine the number of solutions of equation (3) belonging to l_2 .

Let us now consider the operator C_0 , defined as the restriction of the operator C by means of the additional condition $u_0 = 0$. The defect subspace T_{λ}^0 of the operator C_0 consists of the quadratically summable solutions of equation (3). The dimension of the defect subspace T_{λ} of the operator C is determined by

the formula $\dim T_\lambda^0 = \dim T_\lambda + m$, obtained on the basis of a lemma of I. M. Glazman ((⁹), p. 47).

A particular case of J_m -matrices is given by C_m -matrices with numerical coefficients (see Sec. 1). However, instead of system (3), for C_m -matrices it is more convenient to write a single difference equation of order $2m$

$$\sum_{j=k-m}^{k+m} c_{kj} x_j = \bar{\lambda} x_k.$$

The corresponding characteristic equation has the form

$$\sum_{s=0}^{2m} f_s(k) w^{2m-s}(k) = 0, \quad (5)$$

where $f_s(k) = c_{k,k+m-s}/c_{k,k+m}$ for $s \neq m$, and $f_m^{(k)} = (c_{kk} - \bar{\lambda})/c_{k,k+m}$. We number the roots of equation (5) in increasing order of their moduli and introduce the numbers $n_\pm^\nu$ and $p_\pm$ exactly as was indicated above.

Theorem 4. Suppose the elements of the C_m -matrix are such that:

1)

$$\sum_{k=0}^{\infty} |f_s(k+1) - f_s(k)| < \infty \quad \text{for } s = 1, 2, \dots, 2m;$$

2) the limiting values of all $2m$ roots $w_j = \lim_{k \rightarrow \infty} w_j(k)$ of equation (5) are nonzero and distinct;

3) there exists $\nu > 1/2$ such that $n_\pm^\nu + p_\pm = 2m$.

Then the deficiency indices of the operator $\tilde{C}$ are $(m - p_+, m - p_-)$.

As examples, let us consider operators C generated by C_m -matrices with elements of the form $c_{k,k+j} = a_j k^{\alpha_j}$ ($j = 0, 1, \dots, m$). If $\alpha_0 = \alpha_m > \alpha_i + 1$ ($i = 1, 2, \dots, m-1$) and $2|a_m| > |a_0|$, then C has deficiency indices (m, m) , although it does not satisfy the conditions of Theorem 2. As an example of an operator with deficiency indices (ρ, ρ) , where $0 < \rho < m$, one may take the case: $m = 2$, $c_{kk} = (a + 2a/k)k^\alpha$, $c_{k,k+1} = bk^\alpha$, $c_{k,k+2} = k^\alpha$, $\alpha > 1$, $|a + 2| < 2|b|$, $a < b^2/4 + 2$. The corresponding operator C has deficiency indices $(1, 1)$.

For ordinary Jacobi matrices the following theorem holds.

Theorem 5. Let C be the operator generated by a Jacobi matrix with elements $a_k = ak^\alpha$ and $b_k = bk^\beta$ (a and α are real, $\beta > 1$, b is complex). Then: 1) C is self-adjoint if $\alpha > \beta$, or if $\alpha = \beta$ and $|a| \geq 2|b|$; 2) C has deficiency indices $(1, 1)$ if $\alpha < \beta$, or if $\alpha = \beta$ and $|a| < 2|b|$.

5. The results presented find application in clarifying the question of self-adjointness of certain operators in the space of second quantization. Let, as usual, $a^*(p)$ and $a(p)$ be the creation and annihilation operators satisfying the Bose or Fermi commutation relations. The spaces H_k are symmetrized or antisymmetrized tensor products of k copies of the space $L_2(E_3)$, where E_3 is three-dimensional Euclidean space. With the aid of Theorem 1, for example, one verifies the self-adjointness of the operator $T = T_0 + V + V^*$, where

$$T_0 = \int_{E_3} \omega(p) a^*(p) a(p) dp + \int_{E_3} \int_{E_3} \varphi(p_1, p_2) a^*(p_1) a^*(p_2) a(p_1) a(p_2) dp_1 dp_2,$$

$$V = \int_{E_3} \int_{E_3} \int_{E_3} \int_{E_3} v(p_1, p_2, p_3 | q) a^*(p_1) a^*(p_2) a^*(p_3) a(q) d^3p dq,$$

$$\omega(p) \geq 0, \quad \varphi(p_1, p_2) \geq \delta > 0, \quad v \in L_2(E_{12}), \quad \|v\| \delta^{-1} < 1/2.$$

When considering the model of a system with one degree of freedom, i.e., when the argument p assumes a single value, one can use Theorem 4. For example, the operator C generated by the expression

$$a^*a + \varepsilon(\bar{\gamma}_2 a^{*4} + \bar{\gamma}_1 a^{*3}a + \gamma_0 a^{*2}a^2 + \gamma_1 a^*a^3 + \gamma_2 a^4)$$

is self-adjoint for arbitrary real ε and γ_0 , if the roots of the equation

$$\gamma_2 w^4 + \gamma_1 w^3 + \gamma_0 w^2 + \gamma_1 w + \bar{\gamma}_2 = 0$$

are distinct and different from zero, and the moduli of two of them are less than 1 and of two are greater than 1. This holds if $|\gamma_0| \geq 2|\gamma_1| + 2|\gamma_2|$. The deficiency indices of such an operator will be (2, 2), if $\gamma_1 = 0$ and $|\gamma_0| < 2|\gamma_2|$.

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Note: Figure translations are in progress. See original paper for figures.

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