

MUTUAL RESTRICTIONS ON CURVATURE AND AREA FOR SURFACES LYING IN A COMPACT PART OF SPACE

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Abstract

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UDC 513

MATHEMATICS

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MUTUAL RESTRICTIONS ON CURVATURE AND AREA FOR SURFACES LYING IN A COMPACT PART OF SPACE

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It is natural to expect that if a closed surface of fixed area is contained in a small ball, then it has many folds and protrusions; more precisely, the variation of its Gaussian curvature is large. The results given below reflect this intuitive fact to some extent.

Let us introduce the following notation: F is a surface of class C^2 , C^2 -isometrically immersed in E^3 ; S is the area of F ; p is the length of the boundary of F ; d is the diameter of a ball containing F . Put

$$M = \int_F |K| dS, \quad \omega^+ = \int_F K^+ dS,$$

where K is the Gaussian curvature of F , and the integration is with respect to surface area. Absolute positive constants will be denoted by the letter C with various subscripts. Admissible (not best possible) values of these constants are established in the course of the proofs.

Theorem 1. *If F is a closed surface, then*

$$S \leq C_1 M d^2. \quad (1)$$

It is clear that the right-hand side in (1) is equal to $2C_1(\omega^+ - \pi\chi)d^2$, where χ is the Euler characteristic of F . Inequality (1) may be regarded as a two-dimensional analogue of the well-known estimate of the length of a curve in terms of its diameter and the variation of its rotation ⁽¹⁾.

Theorem 2. *If the surface F is homeomorphic to a disk and $\omega^+ < \pi$, then*

$$S \leq (C_2 + C_3 M)^{3/2} (p\sqrt{pd} + pd). \quad (2)$$

Corollary. *If the surface F is a simply connected geodesic disk of radius r and $\omega^+ < \pi$, then*

$$r \leq (C_4 + C_5 M)^6 d.$$

Therefore a complete surface homeomorphic to the plane, for which $\omega^+ < \pi$, $M < \infty$, is unbounded in E^3 .

Remark. The requirement $\omega^+ < \pi$ is apparently not essential. The following simple theorem speaks in favor of this conjecture.

Theorem 3. *Let F be a locally convex surface (not necessarily of class C^2). Then*

$$S \leq C_6(2 + \omega^+ - \chi)^3 d(p + d).$$

The proofs of Theorems 1-3 are obtained by a limiting transition from analogous theorems for polyhedral surfaces. These latter theorems seem to us to be of independent interest. We give their statements.

By a polyhedral surface $F = (R, \varphi)$ we mean an isometric immersion φ of a two-dimensional manifold R with a polyhedral metric into E^3 (or into E^2), under which every triangle of some

* What is meant is homeomorphism to a disk of the abstractly given surface, and not of the corresponding set in E^3 .

of the triangulation T of the manifold R pass into a flat rectilinear triangle.

Let us introduce the positive and negative external curvatures μ^+, μ^- of the polyhedral surface F . Let $X \in \text{int } R$, and let U_X be a neighborhood of X in R consisting of the triangles of the triangulation T adjacent to X . Denote by $P_X(\nu)$ the plane passing through the point $\varphi(X)$ with normal ν , and let $k(X, \nu)$ be the number of components of the set $\varphi(U_X) \setminus P_X(\nu)$. Put

$$n^+(\nu) = \sum_{k(X, \nu)=1} k(X, \nu), \quad n^-(\nu) = \sum_{k(X, \nu)>2} \left(\frac{1}{2} k(X, \nu) - 1 \right),$$

$$\mu^+ = \frac{1}{2} \int n^+(\nu) d\sigma_\nu, \quad \mu^- = \frac{1}{2} \int n^-(\nu) d\sigma_\nu,$$

where the integration is over the unit sphere.

It can be shown that the difference $\mu^+ - \mu^-$ is equal to the intrinsic curvature of F . Put, for the polyhedral surface, $M = \mu^+ + \mu^-$. Then the following assertions hold.

Theorem 1'. If F is a closed polyhedral surface, then estimate (1) holds.

Theorem 2'. If $F(R, \varphi)$ is a polyhedral surface, where the manifold R is homeomorphic to a disk and $\mu^+ < \pi$, then estimate (2) holds.

Remark. Let R be a closed two-dimensional manifold with a polyhedral metric, and let $\{F_n\}$ be a sequence of polyhedral surfaces $F_n = (R, \varphi_n)$ isometric to one another, with $\text{diam } \varphi_n(R) \rightarrow 0$ (such a sequence always exists ⁽²⁾). Then it follows from Theorem 1' that $\mu^+(F_n) \rightarrow \infty$.

The proof of Theorem 1' is first carried out for specialized polyhedral surfaces in E^2 and is extended to the general case by the method of projections ⁽³⁾. The polyhedral surfaces used are characterized by the fact that no interior point of them is adjacent to more than two faces. Here, by the boundary of the surface $F = (R, \varphi)$ in E^2 we mean the maximal open set $Q \subset R$ such that each point $X \in Q$ has a neighborhood U which φ maps isometrically onto the set $\varphi(U)$.

The following plays the main role in the proof.

Lemma 1. Let F be a polyhedral surface in E^2 having only one face. Denote by $\tau(X)$ the turn of ∂R at a point X which is not a singleton component of ∂R . Then the estimate holds

$$S \leq C_4 d^2 \left\{ \sum_{\tau(X) \in [-\pi, \pi/2]} [\pi - \tau(X)] + \sum_{\tau(X) \in [-\pi, \pi/2]} |\tau(X)| \right\}.$$

In the proof of Theorem 2', along with Lemma 1, the following is used.

Lemma 2. Let a geodesic broken line with endpoints A, B be contained in a boundary-convex domain $G \subset R$ homeomorphic to a disk, with $A, B \in \partial G$. Denote by Λ' the plane broken line with endpoints A', B' , whose links are respectively equal to the links of Λ , and whose turns at the vertices of Λ' are equal to the turns at the corresponding vertices of Λ (all turns are taken on one side of the broken line). Then, if $\text{diam } \Lambda' \leq \text{diam } \Lambda = \rho(A, B)$ and the absolute curvature of the domain G is small (it is enough that $\text{var } \omega(G) \leq 0.001$), then $A'B' > 0.1\rho(A, B)$, where ρ is the distance in the metric of R .

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Note: Figure translations are in progress. See original paper for figures.

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